Integrable systems and associative geometry

Alberto Tacchella
altacch@gmail.com

SISSA – December 19, 2017

Outline of the talk

1. Mini-crash course on associative geometry
2. Integrating systems by the projection method
3. Some developments

Geometric objects on associative algebras

In ordinary (commutative) geometry:

- space X, sheaf $\mathcal{O}_X$, commutative algebra $A_X = \Gamma(\mathcal{O}_X)$
- tangent sheaf $\mathcal{T}_X$, vector fields $\mathcal{X}^1 = \Gamma(\mathcal{T}_X)$, polyvector fields $\mathcal{X}^\bullet$
- cotangent sheaf Ω_X, 1-forms $\Omega^1 = \Gamma(\Omega_X)$, differential forms $\Omega^\bullet$

Geometric objects on associative algebras

In ordinary (commutative) geometry:

- space X, sheaf $\mathcal{O}_X$, commutative algebra $A_X = \Gamma(\mathcal{O}_X)$
- tangent sheaf $\mathcal{T}_X$, vector fields $\mathcal{X}^1 = \Gamma(\mathcal{T}_X)$, polyvector fields $\mathcal{X}^\bullet$
- cotangent sheaf Ω_X, 1-forms $\Omega^1 = \Gamma(\Omega_X)$, differential forms $\Omega^\bullet$

In associative geometry:

- ~~space~~ ~~sheaf~~ associative algebra A, vector space $A_\natural = A/[A, A]$
- graded vec.sp. $\mathcal{V}^\bullet(A)$, bracket $[\mathcal{V}^p(A), \mathcal{V}^q(A)] \subseteq \mathcal{V}^{p+q-1}(A)$
- graded vec.sp. $\mathrm{DR}^\bullet(A)$, differential $\mathrm{d}\colon \mathrm{DR}^k(A) \to \mathrm{DR}^{k+1}(A)$

Geometric objects on associative algebras

In ordinary (commutative) geometry:

- space X, sheaf $\mathcal{O}_X$, commutative algebra $A_X = \Gamma(\mathcal{O}_X)$
- tangent sheaf $\mathcal{T}_X$, vector fields $\mathcal{X}^1 = \Gamma(\mathcal{T}_X)$, polyvector fields $\mathcal{X}^\bullet$
- cotangent sheaf Ω_X, 1-forms $\Omega^1 = \Gamma(\Omega_X)$, differential forms $\Omega^\bullet$

In associative geometry:

- ~~space~~ ~~sheaf~~ associative algebra A, vector space $A_\natural = A/[A,A]$
- graded vec.sp. $\mathcal{V}^\bullet(A)$, bracket $[\mathcal{V}^p(A), \mathcal{V}^q(A)] \subseteq \mathcal{V}^{p+q-1}(A)$
- graded vec.sp. $\mathrm{DR}^\bullet(A)$, differential $\mathrm{d}\colon \mathrm{DR}^k(A) \to \mathrm{DR}^{k+1}(A)$

Enough to do:

- *symplectic geometry*: pair (A, ω) with $\omega \in \mathrm{DR}^2(A)$ such that $\mathrm{d}\omega = 0$, map $\theta \mapsto \omega(\theta, -)$ is invertible
- *Poisson geometry*: pair (A, π) with $\pi \in \mathcal{V}^2(A)$ such that $[\pi, \pi] = 0$,
- ...

The route to ordinary manifolds

Family of affine schemes/varieties

$$\mathrm{Rep}_d^A := \mathrm{Hom}_{\mathbb{K}\text{-}\mathbf{Alg}}(A, \mathrm{Mat}_{d,d}(\mathbb{K}))$$

with $\mathrm{GL}_d(\mathbb{K})$-action given by

$$(g.\rho)(a) := g\rho(a)g^{-1}$$

Moduli space of representations:

$$\mathcal{R}_d^A := \mathrm{Rep}_d^A /\!\!/ \mathrm{GL}_d(\mathbb{K})$$

The route to ordinary manifolds

Family of affine schemes/varieties

$$\mathrm{Rep}_d^A := \mathrm{Hom}_{\mathbb{K}\text{-}\mathbf{Alg}}(A, \mathrm{Mat}_{d,d}(\mathbb{K}))$$

with $\mathrm{GL}_d(\mathbb{K})$-action given by

$$(g.\rho)(a) := g\rho(a)g^{-1}$$

Moduli space of representations:

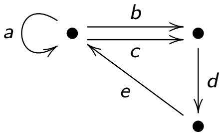
$$\mathcal{R}_d^A := \mathrm{Rep}_d^A /\!\!/ \mathrm{GL}_d(\mathbb{K})$$

Associative-geometric objects on A induce GL_d-invariant objects on Rep_d^A (hence on $\mathcal{R}_d^A$):

$$\begin{aligned} A_\natural &\to \mathbb{K}[\mathrm{Rep}_d^A]^{\mathrm{GL}_d(\mathbb{K})} \\ \mathcal{V}^p(A) &\to \{\mathrm{GL}_d\text{-invariant } p\text{-vector fields on } \mathrm{Rep}_d^A\} \\ \mathrm{DR}^p(A) &\to \{\mathrm{GL}_d\text{-invariant } p\text{-forms on } \mathrm{Rep}_d^A\} \end{aligned}$$

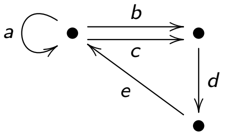
The case of quiver path algebras

A quiver is just another name for a (directed, multi)graph (with loops).



The case of quiver path algebras

A quiver is just another name for a (directed, multi)graph (with loops).

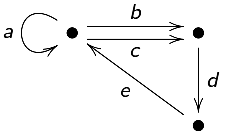


Each quiver Q determines an associative algebra, the path algebra $\mathbb{K}Q$. Generated as a $\mathbb{K}$-vector space by paths (including the trivial ones), with product given by concatenation of paths

$$ba \qquad a^3 \qquad edba \qquad ca^2e \qquad ad = 0 \qquad \ldots$$

The case of quiver path algebras

A quiver is just another name for a (directed, multi)graph (with loops).



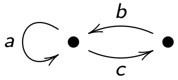
Each quiver Q determines an associative algebra, the path algebra $\mathbb{K}Q$. Generated as a $\mathbb{K}$-vector space by paths (including the trivial ones), with product given by concatenation of paths

$$ba \qquad a^3 \qquad edba \qquad ca^2e \qquad ad = 0 \qquad \ldots$$

We can then apply the above machinery and do (associative) geometry over quivers. This is in fact a particularly nice case, as quiver path algebras are always *formally smooth* ($\Rightarrow \mathrm{Rep}_d^A$ is always smooth).

Fundamentals of associative geometry on $\mathbb{K}Q$

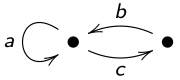
A regular function $f \in A_\natural$ is a sum of necklace words in A, that is cycles in the quiver Q:



$$bca = cab = abc \qquad bcbc = cbcb \qquad \ldots$$

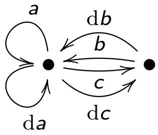
Fundamentals of associative geometry on $\mathbb{K}Q$

A regular function $f \in A_\natural$ is a sum of necklace words in A, that is cycles in the quiver Q:



$$bca = cab = abc \qquad bcbc = cbcb \qquad \dots$$

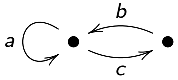
For each arrow $x \in Q$ add a parallel arrow $\mathrm{d}x$. Define $\Omega^p(A)$ as the vector space spanned by the paths in this enlarged quiver with exactly p arrows of the form $\mathrm{d}x$:



$$bc\,\mathrm{d}b \qquad a^2\,\mathrm{d}b\,\mathrm{d}c \qquad \dots$$

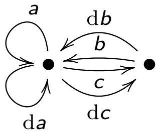
Fundamentals of associative geometry on $\mathbb{K}Q$

A regular function $f \in A_\natural$ is a sum of necklace words in A, that is cycles in the quiver Q:



$$bca = cab = abc \qquad bcbc = cbcb \qquad \dots$$

For each arrow $x \in Q$ add a parallel arrow $\mathrm{d}x$. Define $\Omega^p(A)$ as the vector space spanned by the paths in this enlarged quiver with exactly p arrows of the form $\mathrm{d}x$:

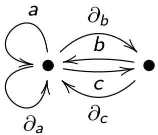


$$bc\,\mathrm{d}b \qquad a^2\,\mathrm{d}b\,\mathrm{d}c \qquad \dots$$

To get $\mathrm{DR}^p(A)$ quotient out the linear subspaces $[\Omega^k(A), \Omega^{p-k}(A)]_{k=0\dots p}$
$\Rightarrow$ the paths which are not closed become zero:

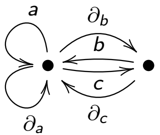
$$bc\,\mathrm{d}b = [bc, \mathrm{d}b] = 0 \in \mathrm{DR}^1(A)$$
$$ca\,\mathrm{d}b = a\,\mathrm{d}b\,c = \mathrm{d}b\,ca \in \mathrm{DR}^1(A)$$
$$a^2\,\mathrm{d}b\,\mathrm{d}c = \mathrm{d}b\,\mathrm{d}c\,a^2 = -\mathrm{d}c\,a^2\,\mathrm{d}b \in \mathrm{DR}^2(A)$$

For each arrow $x \in Q$ add an opposite arrow ∂_x. Define $\mathcal{D}^p(A)$ as the vector space spanned by the paths in this enlarged quiver with exactly p arrows of the form ∂_x:



$$bc\partial_a \qquad c\partial_a\partial_c \qquad \dots$$

For each arrow $x \in Q$ add an opposite arrow ∂_x. Define $\mathcal{D}^p(A)$ as the vector space spanned by the paths in this enlarged quiver with exactly p arrows of the form ∂_x:



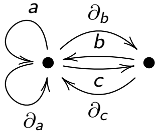
$$bc\partial_a \qquad c\partial_a\partial_c \qquad \dots$$

To get $\mathcal{V}^p(A)$ quotient out the linear subspaces $[\mathcal{D}^k(A), \mathcal{D}^{p-k}(A)]_{k=0\dots p}$
$\Rightarrow$ again, the paths which are not closed become zero:

$$bc\partial_a = c\partial_a b = \partial_a bc \in \mathcal{V}^1(A)$$
$$c\partial_a\partial_c = \partial_a\partial_c c = -\partial_c c\partial_a \in \mathcal{V}^2(A)$$

Fundamentals of associative geometry on $\mathbb{K}Q$

For each arrow $x \in Q$ add an opposite arrow ∂_x. Define $\mathcal{D}^p(A)$ as the vector space spanned by the paths in this enlarged quiver with exactly p arrows of the form ∂_x:



$$bc\partial_a \qquad c\partial_a\partial_c \qquad \ldots$$

To get $\mathcal{V}^p(A)$ quotient out the linear subspaces $[\mathcal{D}^k(A), \mathcal{D}^{p-k}(A)]_{k=0\ldots p}$ $\Rightarrow$ again, the paths which are not closed become zero:

$$bc\partial_a = c\partial_a b = \partial_a bc \in \mathcal{V}^1(A)$$
$$c\partial_a\partial_c = \partial_a\partial_c c = -\partial_c c\partial_a \in \mathcal{V}^2(A)$$

The pairing between a 1-form $\alpha = \sum_{x\in Q} r_x \, \mathrm{d}x$ and a vector field $\theta = \sum_{x\in Q} p_x \partial_x$ is then given by

$$\langle \alpha, \theta \rangle = \sum_{x\in Q} r_x p_x \in A_\natural$$

Induced objects on representation spaces

$$(A, B, C) \in \mathsf{Rep}^{A}_{(n,r)} = \mathsf{Mat}_{n,n}(\mathbb{K}) \oplus \mathsf{Mat}_{n,r}(\mathbb{K}) \oplus \mathsf{Mat}_{r,n}(\mathbb{K})$$

Induced objects on representation spaces

$$(A, B, C) \in \mathrm{Rep}^{A}_{(n,r)} = \mathrm{Mat}_{n,n}(\mathbb{K}) \oplus \mathrm{Mat}_{n,r}(\mathbb{K}) \oplus \mathrm{Mat}_{r,n}(\mathbb{K})$$

- objects in $\Omega^\bullet(A)$ and $\mathcal{D}^\bullet(A)$ induce "matrix-valued objects":

$$\begin{aligned}
&p = bca \in A && \hat{p}(A, B, C) = BCA \\
&\alpha = bc\,\mathrm{d}b \in \Omega^1(A) && \hat{\alpha}(A, B, C) = BC\mathrm{d}B \\
&\omega = a^2\,\mathrm{d}b\,\mathrm{d}c \in \Omega^2(A) && \hat{\omega}(A, B, C) = A^2\mathrm{d}B \wedge \mathrm{d}C \\
&\theta = bc\partial_a \in \mathcal{D}^1(A) && \hat{\theta}(A, B, C) = BC\tfrac{\partial}{\partial A} \\
&\pi = a\partial_c\partial_b \in \mathcal{D}^2(A) && \hat{\pi}(A, B, C) = A\tfrac{\partial}{\partial C} \wedge \tfrac{\partial}{\partial B}
\end{aligned}$$

Induced objects on representation spaces

$$(A, B, C) \in \mathrm{Rep}^A_{(n,r)} = \mathrm{Mat}_{n,n}(\mathbb{K}) \oplus \mathrm{Mat}_{n,r}(\mathbb{K}) \oplus \mathrm{Mat}_{r,n}(\mathbb{K})$$

- objects in $\Omega^\bullet(A)$ and $\mathcal{D}^\bullet(A)$ induce "matrix-valued objects":

$$p = bca \in A \qquad \hat{p}(A, B, C) = BCA$$
$$\alpha = bc\,\mathrm{d}b \in \Omega^1(A) \qquad \hat{\alpha}(A, B, C) = BC\mathrm{d}B$$
$$\omega = a^2\,\mathrm{d}b\,\mathrm{d}c \in \Omega^2(A) \qquad \hat{\omega}(A, B, C) = A^2\mathrm{d}B \wedge \mathrm{d}C$$
$$\theta = bc\partial_a \in \mathcal{D}^1(A) \qquad \hat{\theta}(A, B, C) = BC\frac{\partial}{\partial A}$$
$$\pi = a\partial_c\partial_b \in \mathcal{D}^2(A) \qquad \hat{\pi}(A, B, C) = A\frac{\partial}{\partial C} \wedge \frac{\partial}{\partial B}$$

- the passage to $\mathrm{DR}^\bullet(A)$ and $\mathcal{V}^\bullet(A)$ corresponds to "taking traces":

$$p = bca \in A_\natural \qquad \hat{\hat{p}}(A, B, C) = \mathrm{tr}\, BCA \in \mathbb{K}[\mathrm{Rep}^A_d]$$
$$\alpha = bc\,\mathrm{d}b \in \mathrm{DR}^1(A) \qquad \hat{\hat{\alpha}}(A, B, C) = \mathrm{tr}\, BC\mathrm{d}B \in \Omega^1(\mathrm{Rep}^A_d)$$
$$\omega = a^2\,\mathrm{d}b\,\mathrm{d}c \in \mathrm{DR}^2(A) \qquad \hat{\hat{\omega}}(A, B, C) = \mathrm{tr}\, A^2\mathrm{d}B \wedge \mathrm{d}C \in \Omega^2(\mathrm{Rep}^A_d)$$
$$\theta = bc\partial_a \in \mathcal{V}^1(A) \qquad \hat{\hat{\theta}}(A, B, C) = \mathrm{tr}\, BC\frac{\partial}{\partial A} \in \mathcal{X}^1(\mathrm{Rep}^A_d)$$
$$\pi = a\partial_c\partial_b \in \mathcal{V}^2(A) \qquad \hat{\hat{\pi}}(A, B, C) = \mathrm{tr}\, A\frac{\partial}{\partial C} \wedge \frac{\partial}{\partial B} \in \mathcal{X}^2(\mathrm{Rep}^A_d)$$

Bonus: all these quantities are automatically GL-invariant.

Dynamical systems on a quiver

Definition

A *dynamical system on a quiver* Q is an element of $\mathcal{V}^1(\mathbb{K}Q)$ (that is, a derivation $\mathbb{K}Q \to \mathbb{K}Q$).

Every dynamical system θ on Q induces a family of GL_d-invariant global vector fields on representation spaces of $A = \mathbb{K}Q$:

$$\hat{\hat{\theta}}_d \in \mathcal{X}^1(\mathrm{Rep}_d^A)$$

whose flows are obtained by solving a system of matrix ODEs.

Dynamical systems on a quiver

Definition

A *dynamical system on a quiver Q* is an element of $\mathcal{V}^1(\mathbb{K}Q)$ (that is, a derivation $\mathbb{K}Q \to \mathbb{K}Q$).

Every dynamical system θ on Q induces a family of GL_d-invariant global vector fields on representation spaces of $A = \mathbb{K}Q$:

$$\hat{\hat{\theta}}_d \in \mathcal{X}^1(\mathrm{Rep}_d^A)$$

whose flows are obtained by solving a system of matrix ODEs.
If we have a symplectic or Poisson structure on A then each regular function $H \in A_\natural$ automatically determines a corresponding "Hamiltonian derivation" θ_H given by

$$i_{\theta_H}(\omega) = -\mathrm{d}H \qquad \text{resp.} \qquad \pi(\mathrm{d}H, -)$$

$\Rightarrow$ can speak of "Hamiltonian systems" on Q.

The projection method

A method to integrate Hamiltonian systems using the symplectic reduction of larger (and easier to solve) systems.

- Olshanetsky and Perelomov, Invent. Math. 37 (1976)
- Kazhdan, Kostant and Sternberg, CPAM 31 (1978)
- Olshanetsky and Perelomov, Phys. Rep. 71 (1981)

The projection method

A method to integrate Hamiltonian systems using the symplectic reduction of larger (and easier to solve) systems.

- Olshanetsky and Perelomov, Invent. Math. 37 (1976)
- Kazhdan, Kostant and Sternberg, CPAM 31 (1978)
- Olshanetsky and Perelomov, Phys. Rep. 71 (1981)

Natural reinterpretation in associative geometry: given a derivation θ on A,

$$\begin{matrix}\text{Integration of} \\ \hat{\theta}_d \text{ on } \operatorname{Rep}_d^A\end{matrix} \rightsquigarrow \begin{matrix}\text{Integration of} \\ \hat{\hat{\theta}}_d \text{ on } \mathcal{R}_d^A\end{matrix}$$

Need to reduce also the symplectic (Poisson, bihamiltonian...) structure along with the flow. This may complicate the construction of the manifold $\mathcal{R}_d^A$; in the simplest situations it will be an ordinary (Marsden-Weinstein) symplectic quotient.

Example: rational CM system

Take Q quiver with two loops, $A = \mathbb{C}\langle x, y\rangle$, $\omega = \mathrm{d}y\,\mathrm{d}x$. *Free motion* on A is given by $H = \frac{1}{2}y^2$, so that $\theta_H = y\partial_x$. Flow on $\mathrm{Rep}_d^A \simeq T^*\,\mathrm{Mat}_{d,d}(\mathbb{C})$ is simply

$$X(t) = Y(0)t + X(0)$$

Example: rational CM system

Take Q quiver with two loops, $A = \mathbb{C}\langle x, y\rangle$, $\omega = \mathrm{d}y\,\mathrm{d}x$. *Free motion* on A is given by $H = \frac{1}{2}y^2$, so that $\theta_H = y\partial_x$. Flow on $\operatorname{Rep}_d^A \simeq T^* \operatorname{Mat}_{d,d}(\mathbb{C})$ is simply

$$X(t) = Y(0)t + X(0)$$

Now we select a (co)adjoint orbit in $\mathfrak{gl}_d(\mathbb{C})$ and perform a symplectic quotient via the momentum map

$$\mu\colon \operatorname{Rep}_d^A \to \mathfrak{gl}_d(\mathbb{C}) \qquad \mu(X, Y) = [X, Y]$$

Let $\mathbb{O}_1 :=$ adjoint orbit of minimal dimension in $\mathfrak{gl}_d(\mathbb{C})$. Then

$$\mathcal{C}_d := \mu^{-1}(\mathbb{O}_1)/\operatorname{GL}_d(\mathbb{C})$$

is a smooth symplectic manifold of dimension $2d$.

Example: rational CM system

Take Q quiver with two loops, $A = \mathbb{C}\langle x, y\rangle$, $\omega = \mathrm{d}y\,\mathrm{d}x$. *Free motion* on A is given by $H = \frac{1}{2}y^2$, so that $\theta_H = y\partial_x$. Flow on $\operatorname{Rep}_d^A \simeq T^* \operatorname{Mat}_{d,d}(\mathbb{C})$ is simply

$$X(t) = Y(0)t + X(0)$$

Now we select a (co)adjoint orbit in $\mathfrak{gl}_d(\mathbb{C})$ and perform a symplectic quotient via the momentum map

$$\mu\colon \operatorname{Rep}_d^A \to \mathfrak{gl}_d(\mathbb{C}) \qquad \mu(X, Y) = [X, Y]$$

Let $\mathbb{O}_1 :=$ adjoint orbit of minimal dimension in $\mathfrak{gl}_d(\mathbb{C})$. Then

$$\mathcal{C}_d := \mu^{-1}(\mathbb{O}_1)/\operatorname{GL}_d(\mathbb{C})$$

is a smooth symplectic manifold of dimension $2d$. The open dense subset where X is diagonalizable is symplectomorphic to $T^*\mathbb{C}^{(d)}$ and the above flow corresponds to the flow determined by

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{1}{(q_i - q_j)^2}$$

Example: rational CM system

If we select a bigger adjoint orbit in $\mathfrak{gl}_d(\mathbb{C})$: singular simplectic reduction (Hochgerner) leads to Calogero-Moser systems with "spin variables"

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{\lambda_{ij}^2}{(q_i - q_j)^2}$$

Example: rational CM system

If we select a bigger adjoint orbit in $\mathfrak{gl}_d(\mathbb{C})$: singular simplectic reduction (Hochgerner) leads to Calogero-Moser systems with "spin variables"

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{\lambda_{ij}^2}{(q_i - q_j)^2}$$

More generally we could take $H = \frac{1}{2}y^2 + p(x)$, with p polynomial which plays the role of "external potential" after the reduction step. One then needs to solve the matrix ODE

$$\ddot{X} + p'(X) = 0$$

which is possible e.g. when $p(x) = \omega^2 x^2$ (harmonic potential), obtaining

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\left(\frac{1}{(q_i - q_j)^2} + \omega^2 (q_i - q_j)^2\right)$$

Example: trigonometric CM system

Now take (A, π) where $\pi = y\partial_y\partial_y + x\partial_y\partial_x$. On the open subset $\mathcal{U} \subset \mathsf{Rep}_d^A$ where X is invertible the induced Poisson bivector $\hat{\pi}$ is non degenerate. The corresponding symplectic form reads

$$\omega_{(X,Y)} = \operatorname{tr}(\mathrm{d}Y \wedge X^{-1}\mathrm{d}X - YX^{-1}\mathrm{d}X \wedge X^{-1}\mathrm{d}X)$$

which is the canonical form on $T^* \operatorname{GL}_d(\mathbb{C})$ if we identify this (trivial) bundle with $\mathcal{U} = \operatorname{GL}_d(\mathbb{C}) \times \operatorname{Mat}_{d,d}(\mathbb{C})$ by means of left translations.

Example: trigonometric CM system

Now take (A, π) where $\pi = y\partial_y\partial_y + x\partial_y\partial_x$. On the open subset $\mathcal{U} \subset \mathsf{Rep}_d^A$ where X is invertible the induced Poisson bivector $\hat{\pi}$ is non degenerate. The corresponding symplectic form reads

$$\omega_{(X,Y)} = \mathrm{tr}(\mathrm{d}Y \wedge X^{-1}\mathrm{d}X - YX^{-1}\mathrm{d}X \wedge X^{-1}\mathrm{d}X)$$

which is the canonical form on $T^* \mathrm{GL}_d(\mathbb{C})$ if we identify this (trivial) bundle with $\mathcal{U} = \mathrm{GL}_d(\mathbb{C}) \times \mathrm{Mat}_{d,d}(\mathbb{C})$ by means of left translations. We perform again a symplectic quotient with respect to the action of $\mathrm{GL}_d(\mathbb{C})$, but this time using the momentum map

$$\mu \colon \mathcal{U} \to \mathfrak{gl}_d(\mathbb{C}) \qquad \mu(X, Y) = XYX^{-1} - Y$$

We obtain another smooth symplectic manifold of dimension $2d$

$$\mathcal{C}_d^{tr} := \mu^{-1}(\mathbb{O}_1)/\,\mathrm{GL}_d(\mathbb{C})$$

Example: trigonometric CM system

Taking again $H = \frac{1}{2}y^2$, the projection of the flow on the open dense subset of $\mathcal{C}_d^{tr}$ where X is diagonalizable corresponds to the flow determined by

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i \neq j}(\sin(q_i - q_j))^{-2} \qquad \text{(and/or sinh)}$$

Reducing on bigger adjoint orbits: trigonometric CM systems with spin.

Example: trigonometric CM system

Taking again $H = \frac{1}{2}y^2$, the projection of the flow on the open dense subset of $\mathcal{C}_d^{tr}$ where X is diagonalizable corresponds to the flow determined by

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i \neq j}(\sin(q_i - q_j))^{-2} \qquad \text{(and/or sinh)}$$

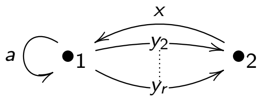
Reducing on bigger adjoint orbits: trigonometric CM systems with spin.
We can also consider the "dual" open dense subset of $\mathcal{C}_d^{tr}$ where Y is diagonalizable. The flows determined on this subset by the functions $H_k = x^k$ coincide with the flows determined by the *light-cone Hamiltonans* for the rational Ruijsenaars-Schneider system:

$$S_k = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = k}} \mathrm{e}^{\sum_{i \in I} p_i} \prod_{\substack{i \in I \\ j \notin I}} \sqrt{1 + \frac{g^2}{(x_i - x_j)^2}}$$

This is related to the Ruijsenaars duality which holds between these two systems.

Example: GH system

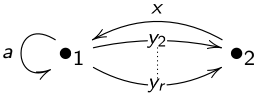
Take $A = \mathbb{C}\boldsymbol{Q}_r$, where $\boldsymbol{Q}_r$ is the double of the quiver



with canonical symplectic form $\omega = \mathrm{d}a^*\mathrm{d}a + \mathrm{d}x^*\mathrm{d}x + \sum_{i=2}^{r} \mathrm{d}y_i^*\mathrm{d}y_i$.

Example: GH system

Take $A = \mathbb{C}\boldsymbol{Q}_r$, where $\boldsymbol{Q}_r$ is the double of the quiver

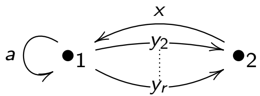


with canonical symplectic form $\omega = \mathrm{d}a^*\mathrm{d}a + \mathrm{d}x^*\mathrm{d}x + \sum_{i=2}^r \mathrm{d}y_i^*\mathrm{d}y_i$.
Points in $\mathrm{Rep}^A_{(n,1)}$ are tuples $(X, Y, v_1, w_2, \ldots, w_r, w_1, v_2, \ldots, v_r)$; the relevant momentum map is

$$\mu(X, Y, v_\alpha, w_\alpha) = [X, Y] + v_1 w_1 - \sum_{i=2}^{r} v_i w_i$$

Example: GH system

Take $A = \mathbb{C}\boldsymbol{Q}_r$, where $\boldsymbol{Q}_r$ is the double of the quiver



with canonical symplectic form $\omega = \mathrm{d}a^* \mathrm{d}a + \mathrm{d}x^* \mathrm{d}x + \sum_{i=2}^r \mathrm{d}y_i^* \mathrm{d}y_i$.
Points in $\mathrm{Rep}^A_{(n,1)}$ are tuples $(X, Y, v_1, w_2, \ldots, w_r, w_1, v_2, \ldots, v_r)$; the relevant momentum map is

$$\mu(X, Y, v_\alpha, w_\alpha) = [X, Y] + v_1 w_1 - \sum_{i=2}^{r} v_i w_i$$

The flow induced by the Hamiltonian $\frac{1}{2}a^{*2}$ on $\mathrm{Rep}^A_{(d,1)}$ is

$$\Phi_t(X, Y, v_\alpha, w_\alpha) = (X + tY, Y, v_\alpha, w_\alpha)$$

On the quotient $\mu^{-1}(\tau I) /\!\!/ \mathrm{GL}_d(\mathbb{C})$ it reduces to

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{\tau^2}{2}\sum_{i \neq j} \frac{\langle f_i, e_j \rangle \langle f_j, e_i \rangle}{(q_i - q_j)^2}$$

Associative bihamiltonian structures

A bihamiltonian manifold is a manifold M which admits two distinct Poisson bivectors π_0, π_1 such that $[\pi_0, \pi_1] = 0$. (Equivalently: $\pi_0 + \pi_1$ is Poisson, $\pi_0 + \lambda\pi_1$ is Poisson for every $\lambda \in \mathbb{P}^1$.)

Associative bihamiltonian structures

A bihamiltonian manifold is a manifold M which admits two distinct Poisson bivectors π_0, π_1 such that $[\pi_0, \pi_1] = 0$. (Equivalently: $\pi_0 + \pi_1$ is Poisson, $\pi_0 + \lambda\pi_1$ is Poisson for every $\lambda \in \mathbb{P}^1$.)
The translation to the associative realm is straightforward: triple (A, π_0, π_1) with $\pi_0, \pi_1 \in \mathcal{V}^2(A)$ double Poisson structures such that

$$[\pi_0, \pi_1] = 0 \in \mathcal{V}^3(A)$$

Associative bihamiltonian structures

A bihamiltonian manifold is a manifold M which admits two distinct Poisson bivectors π_0, π_1 such that $[\pi_0, \pi_1] = 0$. (Equivalently: $\pi_0 + \pi_1$ is Poisson, $\pi_0 + \lambda\pi_1$ is Poisson for every $\lambda \in \mathbb{P}^1$.)
The translation to the associative realm is straightforward: triple (A, π_0, π_1) with $\pi_0, \pi_1 \in \mathcal{V}^2(A)$ double Poisson structures such that

$$[\pi_0, \pi_1] = 0 \in \mathcal{V}^3(A)$$

Classical method to manufacture bihamiltonian structures: π_0 comes from a symplectic form, π_1 is built from π_0 by means of a *recursion operator* $N\colon TM \to TM$ whose Nijenhuis torsion vanishes:

$$\mathcal{T}_N(X, Y) := [N(X), N(Y)] - N\big([N(X), Y] + [X, N(Y)] - N([X, Y])\big)$$

Associative bihamiltonian structures

A bihamiltonian manifold is a manifold M which admits two distinct Poisson bivectors π_0, π_1 such that $[\pi_0, \pi_1] = 0$. (Equivalently: $\pi_0 + \pi_1$ is Poisson, $\pi_0 + \lambda\pi_1$ is Poisson for every $\lambda \in \mathbb{P}^1$.)
The translation to the associative realm is straightforward: triple (A, π_0, π_1) with $\pi_0, \pi_1 \in \mathcal{V}^2(A)$ double Poisson structures such that

$$[\pi_0, \pi_1] = 0 \in \mathcal{V}^3(A)$$

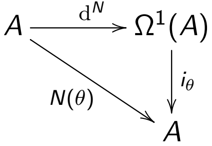
Classical method to manufacture bihamiltonian structures: π_0 comes from a symplectic form, π_1 is built from π_0 by means of a *recursion operator* $N\colon TM \to TM$ whose Nijenhuis torsion vanishes:

$$\mathcal{T}_N(X, Y) := [N(X), N(Y)] - N\big([N(X), Y] + [X, N(Y)] - N([X, Y])\big)$$

In the associative realm, the recursion operator should be given by a $\mathbb{K}$-linear map $N\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$. Need some condition to ensure that the result of N "depends linearly" on the source derivation...

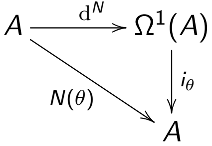
Associative bihamiltonian structures

Let us call a map $N\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ *regular* if there exists a derivation $\mathrm{d}^N\colon A \to \Omega^1(A)$ such that, for every $\theta \in \mathcal{V}^1(A)$, the derivation $N(\theta)$ factorizes as



Associative bihamiltonian structures

Let us call a map $N\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ *regular* if there exists a derivation $\mathrm{d}^N\colon A \to \Omega^1(A)$ such that, for every $\theta \in \mathcal{V}^1(A)$, the derivation $N(\theta)$ factorizes as



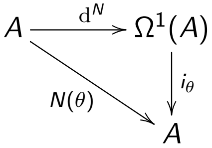
This implies in particular that the *transpose* of N is well defined as the unique map $N^*\colon \mathrm{DR}^1(A) \to \mathrm{DR}^1(A)$ such that

$$\langle N^*(\alpha), \theta\rangle = \langle \alpha, N(\theta)\rangle$$

(In fact d^N induces a whole "deformed Cartan calculus" on $\mathrm{DR}^\bullet(A)$...)

Associative bihamiltonian structures

Let us call a map $N\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ *regular* if there exists a derivation $\mathrm{d}^N\colon A \to \Omega^1(A)$ such that, for every $\theta \in \mathcal{V}^1(A)$, the derivation $N(\theta)$ factorizes as



This implies in particular that the *transpose* of N is well defined as the unique map $N^*\colon \mathrm{DR}^1(A) \to \mathrm{DR}^1(A)$ such that

$$\langle N^*(\alpha), \theta\rangle = \langle \alpha, N(\theta)\rangle$$

(In fact d^N induces a whole "deformed Cartan calculus" on $\mathrm{DR}^\bullet(A)$...)
Now we can simply define a Nijenhuis tensor on A as a regular map $N\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ such that $\mathcal{T}_N = 0$.

Associative bihamiltonian structures

So we have a notion of Nijenhuis tensor on an associative algebra A. If some further compatibility conditions between π_0 and N are satisfied, we speak of a Poisson-Nijenhuis structure on Q.

Theorem (C. Bartocci, A.T, LMP 2017)

Let Q be a quiver and (π, N) a Poisson-Nijenhuis structure on it. Then the bivector

$$\pi^N(\alpha, \beta) := \pi(N^*(\alpha), \beta) = \pi(\alpha, N^*(\beta))$$

is a double Poisson structure on A with $[\pi_0, \pi_1] = 0$.

Associative bihamiltonian structures

So we have a notion of Nijenhuis tensor on an associative algebra A. If some further compatibility conditions between π_0 and N are satisfied, we speak of a Poisson-Nijenhuis structure on Q.

Theorem (C. Bartocci, A.T, LMP 2017)

Let Q be a quiver and (π, N) a Poisson-Nijenhuis structure on it. Then the bivector

$$\pi^N(\alpha, \beta) := \pi(N^*(\alpha), \beta) = \pi(\alpha, N^*(\beta))$$

is a double Poisson structure on A with $[\pi_0, \pi_1] = 0$.

As an example, the associative bihamiltonian structure for the rational CM system is given by

$$\pi_0 = \partial_y \partial_x$$
$$\pi_1 = y\partial_y \partial_x + x\partial_x \partial_x$$

with recursion operator $N(\theta)(x, y) = (y\theta(x) + [\theta(y), x], \theta(y)y)$.

Associative complex structures

Nijenhuis torsion is also used to give a notion of integrability for complex structures on a real manifold. Can we find an associative analogue also for this construction?

Associative complex structures

Nijenhuis torsion is also used to give a notion of integrability for complex structures on a real manifold. Can we find an associative analogue also for this construction?

Let Q be a quiver, $A = \mathbb{R}Q$. A regular endomorphism $I \colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ is a *complex structure on* Q provided that:

1. $I^2 = -\operatorname{id}_{\mathcal{V}^1(A)}$,
2. $\mathcal{T}_I = 0$.

Associative complex structures

Nijenhuis torsion is also used to give a notion of integrability for complex structures on a real manifold. Can we find an associative analogue also for this construction?

Let Q be a quiver, $A = \mathbb{R}Q$. A regular endomorphism $I\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ is a *complex structure on* Q provided that:

1. $I^2 = -\operatorname{id}_{\mathcal{V}^1(A)}$,
2. $\mathcal{T}_I = 0$.

Basic example on $\overline{Q}$, where Q is the quiver with one vertex and n loops:

$$I(\theta)(a_1, \dots, a_n, a_1^*, \dots, a_n^*) = (-\theta(a_1^*), \dots, -\theta(a_n^*), \theta(a_1), \dots, \theta(a_n))$$

This recovers the complex structure on $\operatorname{Rep}_d^{\mathbb{R}\overline{Q}} \simeq T\operatorname{Mat}_{d,d}(\mathbb{R})^{\oplus n}$ which corresponds to the complex vector space $\operatorname{Mat}_{d,d}(\mathbb{C})^{\oplus n}$.

Clearly, more non-trivial examples are needed...

Associative Kähler manifolds

Classically: quadruple (M, I, g, ω) with M (real) manifold, I integrable complex structure, g Riemannian metric, ω symplectic form which satisfy a number of compatibility conditions.

Associative Kähler manifolds

Classically: quadruple (M, I, g, ω) with M (real) manifold, I integrable complex structure, g Riemannian metric, ω symplectic form which satisfy a number of compatibility conditions.
In our case it is natural to start from the data (A, I, ω) and recover the "Riemannian metric" as

$$g(\theta, \eta) := i_{I(\eta)}(i_\theta(\omega)) \qquad (\theta, \eta \in \mathcal{V}^1(A))$$

then require the symmetry constraint $g(\theta, \eta) - g(\eta, \theta) = 0 \mod [A, A]$.

Associative Kähler manifolds

Classically: quadruple (M, I, g, ω) with M (real) manifold, I integrable complex structure, g Riemannian metric, ω symplectic form which satisfy a number of compatibility conditions.
In our case it is natural to start from the data (A, I, ω) and recover the "Riemannian metric" as

$$g(\theta, \eta) := i_{I(\eta)}(i_\theta(\omega)) \qquad (\theta, \eta \in \mathcal{V}^1(A))$$

then require the symmetry constraint $g(\theta, \eta) - g(\eta, \theta) = 0 \mod [A, A]$.
The basic example should again be the double of the n-loop quiver with I defined as before, $\omega = \sum_i \mathrm{d}a_i^* \mathrm{d}a_i$ and

$$g(\theta, \eta) = \sum_i (\theta(a_i)\eta(a_i) + \eta(a_i^*)\theta(a_i^*))$$

Once the Kähler case is under control, one would like to proceed to the hyper-Kähler case (relevant for Nakajima's quiver varieties).

Associative contact structures

What if the original dynamics depends explicitly on a "time" variable? This situation arises, for instance, in the *noncommutative Painlevé-Calogero correspondence*, where one deals with Hamiltonians such as

$$H_{II} = \frac{1}{2}y^2 - \frac{1}{2}(x^2 + \frac{t}{2})^2 - \theta x$$

(cf. Bertola, Cafasso and Roubtsov, arXiv:1710.00736)

Associative contact structures

What if the original dynamics depends explicitly on a “time” variable? This situation arises, for instance, in the *noncommutative Painlevé-Calogero correspondence*, where one deals with Hamiltonians such as

$$H_{II} = \frac{1}{2}y^2 - \frac{1}{2}(x^2 + \frac{t}{2})^2 - \theta x$$

(cf. Bertola, Cafasso and Roubtsov, arXiv:1710.00736)
Idea: instead of working in $\mathbb{C}Q = \mathbb{C}\langle x, y \rangle$ one considers

$$A := \mathbb{C}\langle x, y, t \rangle / I$$

where $I = (xt - tx, yt - ty)$. The canonical symplectic form on $\mathbb{C}\langle x, y \rangle$ then induces a double Poisson structure on A.

Associative contact structures

What if the original dynamics depends explicitly on a "time" variable? This situation arises, for instance, in the *noncommutative Painlevé-Calogero correspondence*, where one deals with Hamiltonians such as

$$H_{II} = \frac{1}{2}y^2 - \frac{1}{2}(x^2 + \frac{t}{2})^2 - \theta x$$

(cf. Bertola, Cafasso and Roubtsov, arXiv:1710.00736)
Idea: instead of working in $\mathbb{C}Q = \mathbb{C}\langle x, y\rangle$ one considers

$$A := \mathbb{C}\langle x, y, t\rangle / I$$

where $I = (xt - tx, yt - ty)$. The canonical symplectic form on $\mathbb{C}\langle x, y\rangle$ then induces a double Poisson structure on A.
One expects to find a picture similar to the classical case, in which the extended phase space of the system can be seen as a trivial bundle over a line. This should correspond to dealing with associative algebras *over a 1-dimensional polynomial ring*.

Thanks everybody!