Noncommutative symplectic geometry of Gibbons-Hermsen varieties

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Non-commutative symplectic geometry

Quivers and Gibbons-Hermsen varieties

Group actions
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- a (smooth) variety $M$ of even dimension;
- a closed, non-degenerate 2-form $\omega$ on $M$. 
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- Poisson brackets, defined e.g. by $\{f, g\} := \omega(X_f, X_g)$;
- an exact sequence of Lie algebras
  $$0 \to H^0_{\text{dR}}(M) \to \Omega^0(M) \to \mathcal{X}_{\text{symp}}(M) \to H^1_{\text{dR}}(M) \to 0.$$
Founding principle of non-commutative geometry: generalize the well-known dualities of the form

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- might get something different even when \( A \) is commutative!
Some constructions that generalize

- Vector fields as $A$-bimodule derivations $A \to A$;
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- Vector fields as $A$-bimodule derivations $A \to A$;
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- Noncommutative forms $\Omega^\bullet_{nc}(A)$ as elements of the tensor algebra of the $A$-bimodule $\Omega^1_{nc}(A)$;
- Cartan calculus: a degree $-1$ “interior product” $i_\theta$ and a degree $0$ “Lie derivative” $\mathcal{L}_\theta$ satisfying

$$\mathcal{L}_\theta = [\text{d}, i_\theta]$$

$$[\mathcal{L}_\theta, \mathcal{L}_\eta] = \mathcal{L}_{[\theta, \eta]}$$

$$[\mathcal{L}_\theta, i_\gamma] = i_{[\theta, \gamma]}$$
The complex of non-commutative differential forms

Hence we have a complex \((\Omega^\bullet_{nc}(A), d)\). However

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H^k(\Omega^\bullet_{nc}(A)) = \begin{cases} 
\mathbb{K} & \text{if } k = 0 \\
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More interesting is the Karoubi-de Rham complex, defined by

$$\text{DR}^\bullet(A) := \frac{\Omega_{\text{nc}}^\bullet(A)}{[\Omega_{\text{nc}}^\bullet(A), \Omega_{\text{nc}}^\bullet(A)\]}$$

as a quotient of graded vector spaces, with

$$[a, b] = ab - (-1)^{\deg a \deg b} ba$$
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\[
\text{DR}^0(A) = \frac{A}{[A, A]}
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A quiver is just a directed graph with no constraints on loops and multiple arcs.
Quivers by example

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\[
\begin{array}{c}
\circ \rightarrow \circ
\end{array}
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A representation of a quiver is given by the choice of a (f.d.) vector space for each vertex and a linear map for each arrow.

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On \( \text{Rep}(Q, \vec{d}) \) there is the action of an algebraic group \( G_{\vec{d}} \) by “change of basis”.

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[Diagram: A quiver with two arrows pointing from one vertex to another]

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The corresponding quotient,

\[
\mathcal{R}(Q, \vec{d}) = \text{Rep}(Q, \vec{d})/G_{\vec{d}}
\]

is the moduli space of representations of \(Q\) with dim. vector \(\vec{d}\).
Where does symplectic geometry enter the picture?

To each quiver $Q$ we can associate its *double* $Q$, by adding to each arrow $a$ in $Q$ an arrow $a^*$ that goes in the opposite direction.

\[
\begin{array}{c}
\bullet \\
\downarrow & \uparrow \\
\bullet \\
\end{array}
\]

Thus its representation space can be seen as a cotangent bundle:

\[
\text{Rep}(Q, \vec{d}) = T^* \text{Rep}(Q, \vec{d})
\]

and is equipped with a canonical symplectic form. Moreover, the action of $G_{\vec{d}}$ on $\text{Rep}(Q, \vec{d})$ is just the lift of its action on the base, hence it is Hamiltonian and has a moment map $\mu$:

\[
\mu : \text{Rep}(Q, \vec{d}) \rightarrow g_{\vec{d}}
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By judiciously choosing a point in $g_{\vec{d}}$, we can obtain some interesting (smooth, symplectic) varieties via Marsden-Weinstein reduction.
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![Diagram of quiver](image)

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$$\mu : \text{Rep}(\overline{Q}, \vec{d}) \to \mathfrak{g}_{\vec{d}}$$

By judiciously choosing a point in $\mathfrak{g}_{\vec{d}}$, we can obtain some interesting (smooth, symplectic) varieties via Marsden-Weinstein reduction.
Where does \textit{n.c.} symplectic geometry enter the picture?

On the other hand, there is a noncommutative algebra naturally associated with $\overline{Q}$, its \textit{path algebra} $\mathbb{C}\overline{Q}$. It is defined as the $\mathbb{C}$-vector space having as basis the set of \textit{paths} in $\overline{Q}$, with product given by composition of paths (or 0 if the paths do not compose).
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\[
\mathbb{C}\overline{Q} = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}
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A_{11} \simeq \mathbb{C}\langle a, a^*, x^*x \rangle,
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etc.
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etc.

The equivalence class in $\text{DR}^2(\mathbb{C}\overline{Q})$ of the 2-form

$$\omega_{nc} := \sum_{a \in Q} da \, da^*$$

endows $\mathbb{C}\overline{Q}$ with a non-commutative symplectic structure.
Take $r, n \in \mathbb{N}$ and

$$V_{n,r} = \text{Mat}_{n,n}(\mathbb{C}) \oplus \text{Mat}_{n,n}(\mathbb{C}) \oplus \text{Mat}_{n,r}(\mathbb{C}) \oplus \text{Mat}_{r,n}(\mathbb{C})$$

$$= T^\ast(\text{Mat}_{n,n}(\mathbb{C}) \oplus \text{Mat}_{n,r}(\mathbb{C}))$$
Gibbons-Hermsen varieties

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$$= T^*(\text{Mat}_{n,n}(\mathbb{C}) \oplus \text{Mat}_{n,r}(\mathbb{C}))$$

We consider the action of $\text{GL}_n(\mathbb{C})$ on $V_{n,r}$ given by

$$G.(X, Y, v, w) = (GXG^{-1}, GYG^{-1}, Gv, wG^{-1})$$

Hamiltonian action with moment map $V_{n,r} \rightarrow \mathfrak{gl}_n(\mathbb{C})$

$$\mu(X, Y, v, w) = [X, Y] + vw$$
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\]

For every \( \tau \in \mathbb{C}^* \) the action of \( \text{GL}_n(\mathbb{C}) \) on \( \mu^{-1}(\tau I) \) is free, so by Marsden-Weinstein we have a smooth symplectic manifold of dimension \( 2nr \)

\[
C_{n,r} = \mu^{-1}(\tau I) / \text{GL}_n(\mathbb{C})
\]
The $r = 1$ case

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It turns out that in this case we can use a simpler quiver

$$\overline{Q}_\circ = \begin{array}{c}
  a \\
  a^* \\
\end{array}$$

Then $\text{DR}^0(\mathbb{C}\overline{Q}_\circ)$ is the vector space of necklace words in $a$ and $a^*$, with bracket given by

$$\{f, g\} = \frac{\partial f}{\partial a} \frac{\partial g}{\partial a^*} - \frac{\partial f}{\partial a^*} \frac{\partial g}{\partial a} \mod [\mathbb{C}\overline{Q}_\circ, \mathbb{C}\overline{Q}_\circ]$$
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\[ G_{(n,1)} = (\text{GL}_n(\mathbb{C}) \times \text{GL}_1(\mathbb{C}))/\mathbb{C}^* \cong \text{GL}_n(\mathbb{C}) \]

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\[ \overline{Q}_o = \begin{array}{ccc} a & \rightarrow & a^* \\ & \downarrow & \downarrow \\ & \text{•} & \end{array} \]

\[ \mathbb{C}\overline{Q}_o = \mathbb{C}\langle a, a^* \rangle \]

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**Theorem (Ginzburg 2000)**

*Each symplectic variety $\mathcal{C}_{n,1}$ can be embedded as a coadjoint orbit in the Lie algebra $\text{DR}^0(\mathbb{C}\overline{Q}_o)$.***
Calogero-Moser correspondence

Theorem (Berest, Wilson 1999)

The space $\mathcal{C} := \bigsqcup_{n \in \mathbb{N}} \mathcal{C}_{n,1}$ parametrizes isomorphism classes of right ideals in the first Weyl algebra over $\mathbb{C}$

$$A_1 = \mathbb{C}\langle a, a^* \rangle / (aa^* - a^*a - 1)$$

Moreover, the natural action of the group $\text{Aut} A_1$ on each of the $\mathcal{C}_{n,1}$ is transitive.
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This was proved by BW in a very roundabout way. But:

$$\text{Aut} A_1 \simeq \text{Aut}(\mathbb{C}Q_\circ; [a, a^*])$$

the group of symplectic automorphisms of $\mathbb{C}Q_\circ$ (i.e., algebra automorphisms of $\mathbb{C}Q_\circ$ preserving $[a, a^*]$).
The space $\mathcal{C} := \bigsqcup_{n \in \mathbb{N}} \mathcal{C}_{n,1}$ parametrizes isomorphism classes of right ideals in the first Weyl algebra over $\mathbb{C}$.

$$A_1 = \mathbb{C} \langle a, a^* \rangle / (aa^* - a^*a - 1)$$

Moreover, the natural action of the group $\text{Aut } A_1$ on each of the $\mathcal{C}_{n,1}$ is transitive.

This was proved by BW in a very roundabout way. But:

$$\text{Aut } A_1 \simeq \text{Aut}(\overline{\mathbb{C}Q_0}; [a, a^*])$$

the group of symplectic automorphisms of $\overline{\mathbb{C}Q_0}$ (i.e., algebra automorphisms of $\overline{\mathbb{C}Q_0}$ preserving $[a, a^*]$).

So from this perspective it is a natural result after all.
A result of Bielawski and Pidstrigach

Next simplest case: \( r = 2 \). Taking

\[
Q_{BP} = \begin{array}{c}
\bullet \\
\circ \\
\end{array}
\begin{array}{c}
\xrightarrow{a} \\
\xleftarrow{x} \\
\xrightarrow{y} \\
\end{array}
\begin{array}{c}
\bullet \\
\circ \\
\end{array}
\]

we can embed \( V_{n,2} \) into \( \text{Rep}(\overline{Q}_{BP}, (n, 1)) \).
Next simplest case: $r = 2$. Taking

$$Q_{BP} = \begin{array}{ccc}
\bullet & \overset{a}{\Rightarrow} & \bullet \\
\begin{array}{c}
\circlearrowleft \\
\leftarrow x
\end{array} & & \begin{array}{c}
\rightarrow y \\
\end{array}
\end{array}$$

we can embed $V_{n,2}$ into $\text{Rep}(\overline{Q}_{BP}, (n, 1))$.

**Theorem (Bielawski, Pidstrygach 2008)**

The group $\text{TAut}(\mathbb{C}\overline{Q}_{BP}; c)$ of (tame) symplectic automorphisms of $\mathbb{C}\overline{Q}_{BP}$ acts transitively on $C_{n,2}$.

Here $c = [a, a^*] + [x, x^*] + [y, y^*]$.
Some work in progress

It is known from work of Baranowski-Ginzburg-Kustnezev that \( \bigsqcup_{n \in \mathbb{N}} C_{n,r} \) parametrizes isomorphism classes of a certain class of right sub-\( A_1 \)-modules in \( B_1^r \).
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It is known from work of Baranowski-Ginzburg-Kustnezov that \( \bigcup_{n \in \mathbb{N}} C_{n,r} \) parametrizes isomorphism classes of a certain class of right sub-\( A_1 \)-modules in \( B_1^r \).

Conjecture (G. Wilson): on \( C_{n,r} \) there is a transitive action of \( \Gamma_{\text{alg}} \times \text{PGL}_r(\mathbb{C}[z]) \)

(\( \Gamma_{\text{alg}} \) is a group of maps of the form \( e^p I_r \) for some polynomial \( p \in z \mathbb{C}[z] \)).
Non-commutative symplectic geometry
Quivers and Gibbons-Hermsen varieties
Group actions

Some work in progress

It is known from work of Baranowski-Ginzburg-Kustnezev that $\bigcup_{n \in \mathbb{N}} C_{n,r}$ parametrizes isomorphism classes of a certain class of right sub-$A_1$-modules in $B_1^r$.

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$$\Gamma_{\text{alg}} \times \text{PGL}_r(\mathbb{C}[z])$$

(where $\Gamma_{\text{alg}}$ is a group of maps of the form $e^{p}I_r$ for some polynomial $p \in z\mathbb{C}[z]$).

Theorem (A.T., I. Mencattini)

When $r = 2$ there is a morphism of groups

$$i : \Gamma_{\text{alg}} \to \text{PTAut}(\mathbb{C}Q_{BP}; c)$$

and the induced action on $C_{n,2}$ is transitive (at least) on a dense open subset.