Group actions on the moduli spaces of noncommutative instantons

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The Calogero-Moser correspondence

The higher rank case

Work in progress

The varieties $\mathcal{C}_{n,r}$

Take $n, r \in \mathbb{N}$ and the vector space

$$\begin{aligned} V_{n,r} &= \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C}) \oplus \mathrm{Mat}_{r,n}(\mathbb{C}) \\ &= T^*(\mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C})) \end{aligned}$$

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We consider the action of $\mathrm{GL}_n(\mathbb{C})$ on $V_{n,r}$ given by

$$G.(X, Y, v, w) = (GXG^{-1}, GYG^{-1}, Gv, wG^{-1})$$

It is Hamiltonian with moment map $V_{n,r} \to \mathfrak{gl}_n(\mathbb{C})$ given by

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For every $\tau \in \mathbb{C}^*$ the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mu^{-1}(\tau I)$ is free, so by Marsden-Weinstein we have a smooth symplectic manifold of dimension $2nr$

$$\mathcal{C}_{n,r} = \mu^{-1}(\tau I)/\,\mathrm{GL}_n(\mathbb{C})$$

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- as a completion of the phase space of the n-particles, r-components Gibbons-Hermsen integrable system (Wilson);
- as a moduli space of a certain kind of sub-A_1-modules of $(B_1)^r$, where

$$A_1 := \frac{\mathbb{C}\langle x, y\rangle}{(xy - yx - 1)}$$

and B_1 is the localization of A_1 w.r.t. nonzero polynomials (Baranovsky-Ginzburg-Kuznetsov).

The case $r = 1$

The space $\mathcal{C} := \bigsqcup_{n\in\mathbb{N}} \mathcal{C}_{n,1}$ is in bijection with the moduli space of isomorphism classes of right ideals in A_1. Moreover, the natural action of the group $\operatorname{Aut} A_1$ on $\mathcal{C}$ is transitive.

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$$\Phi_p \quad \text{defined by} \quad \begin{cases} x \mapsto x - p'(y) \\ y \mapsto y \end{cases}$$

together with the "formal Fourier transform"

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These generators act on $\mathcal{C}_{n,1}$ as follows:

$$\Phi_p.(X, Y) = (X - p'(Y), Y) \quad \mathcal{F}.(X, Y) = (-Y, X)$$

Relationship with noncommutative geometry

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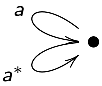
$\overline{Q}_\circ =$ 

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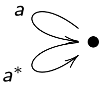
$\overline{Q}_\circ =$ 

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The idea is now to find a suitable family of quivers for which the same construction works when $r > 1$.

A result of Bielawski and Pidstrygach

Next simplest case: $r = 2$. Taking

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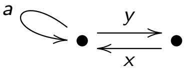

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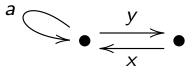

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Let A denote the path algebra of $\overline{Q}_{BP}$; since Q_{BP} has two vertices, A is no longer a free algebra. Denote by $\mathrm{Aut}(A; c)$ the group of n.c. symplectomorphisms of A, i.e. the automorphisms preserving the element

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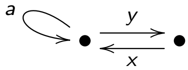
$Q_{BP} =$ 

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Problem: we don't really have a good description of $\mathrm{Aut}(A; c)$...

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$$\begin{pmatrix} -x \\ y^* \end{pmatrix} \mapsto T \begin{pmatrix} -x \\ y^* \end{pmatrix} \qquad \begin{pmatrix} x^* & y \end{pmatrix} \mapsto \begin{pmatrix} x^* & y \end{pmatrix} T^{-1}$$

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Theorem (Bielawski, Pidstrygach 2008)

The group $\mathrm{TAut}(A; c)$ acts transitively on $\mathcal{C}_{n,2}$.

Gibbons-Hermsen flows

The Gibbons-Hermsen Hamiltonians on $\mathcal{C}_{n,r}$ are

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The flow of $J_{k,\alpha}$ is polynomial only when α is either the identity (in which case $J_{k,I} \propto \operatorname{Tr} Y^k$, as a consequence of the moment map equation) or nilpotent (i.e., a multiple of e_{12} or e_{21}).

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For example, one can calculate that the action determined by a linear combination with coefficients $(p_k)_{k \geq 1}$ of the flows of the Hamiltonians $J_{k,e_{21}}$ on $\mathcal{C}_{n,2}$ coincides with the action of the strictly op-triangular automorphism

$$(a, x, y) \mapsto (a - p'(a^*) y^* x^*, x - p(a^*) y^*, y - x^* p(a^*))$$

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$$\{J_{k,\alpha}, J_{\ell,\beta}\} = J_{k+\ell,[\alpha,\beta]}$$

Same relations that hold in the Lie algebra of polynomial loops in $\mathfrak{gl}_r(\mathbb{C})$ (explicitly: $J_{k,\alpha} \leftrightarrow z^k\alpha$). However, we cannot get a group by "integrating" this algebra ($\mathrm{e}^{p_1}\mathrm{e}^{p_2} \neq \mathrm{e}^{p_1+p_2}$).

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Possible solution: use the group Γ of all holomorphic maps $\mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$. But this group is huge and contains non-polynomial flows. Instead, we would like to find a group $\Gamma^{\mathrm{alg}} \subset \Gamma$ that can be embedded in $\mathrm{TAut}(A; c)$.

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Wilson (2009, unpublished) suggests taking

$$\Gamma^{\mathrm{alg}} := \Gamma^{\mathrm{alg}}_{\mathrm{sc}} \times \mathsf{PGL}_2(\mathbb{C}[z])$$

where $\Gamma^{\mathrm{alg}}_{\mathrm{sc}}$ is the group of matrix-valued functions of the form $\mathrm{e}^p I$ for some $p \in z\mathbb{C}[z]$.

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Theorem (I. Mencattini, A.T.)

There is a (nontrivial) morphism of groups

$$i\colon \Gamma^{\mathrm{alg}} \to \mathrm{PTAut}(A; c)$$

Denoting by $\mathcal{P}$ the subgroup of $\mathrm{PTAut}(A; c)$ generated by the image of i and the single symplectic automorphism

$$\mathcal{F}(a, a^*, x, x^*, y, y^*) := (-a^*, a, -y^*, y, -x^*, x)$$

the induced action of $\mathcal{P}$ on $\mathcal{C}_{n,2}$ is transitive (at least) on the open subset of $\mathcal{C}_{n,2}$ where X or Y are regular semisimple.

Proof sketch

Theorem (Nagao)

$\mathrm{GL}_2(\mathbb{C}[z]) \simeq \mathrm{GL}_2(\mathbb{C}) *_{B_2(\mathbb{C})} B_2(\mathbb{C}[z])$

where $B_2(\mathbb{C}[z])$ is the subgroup of lower triangular matrices in $\mathrm{GL}_2(\mathbb{C}[z])$ and $B_2(\mathbb{C})$ is the subgroup of lower triangular matrices in $\mathrm{GL}_2(\mathbb{C})$.

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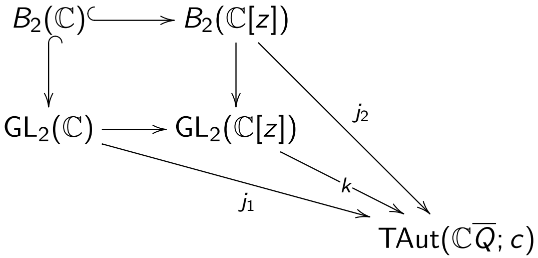

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$$\tilde{k}\colon \mathrm{PGL}_2(\mathbb{C}[z]) \to \mathrm{PTAut}(A; c)$$

We extend $\tilde{k}$ to Γ^{alg} by defining a morphism $j_3\colon \Gamma^{\mathrm{alg}}_{\mathrm{sc}} \to \mathrm{TAut}(A; c)$ which sends $e^p I$ to the automorphism whose only nontrivial action on the generators is

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Finally, i is defined by mapping Γ^{alg} to the internal direct product of $\operatorname{im} j_3$ and $\operatorname{im} \tilde{k}$ in $PTAut(A; c)$.

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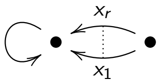

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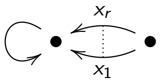

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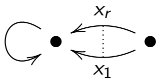

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Ideally: obtain a (transitive?) action of $\Gamma^{\mathrm{alg}}(r)$ on $\mathcal{C}_{n,r}$, for every $r \geq 1$.