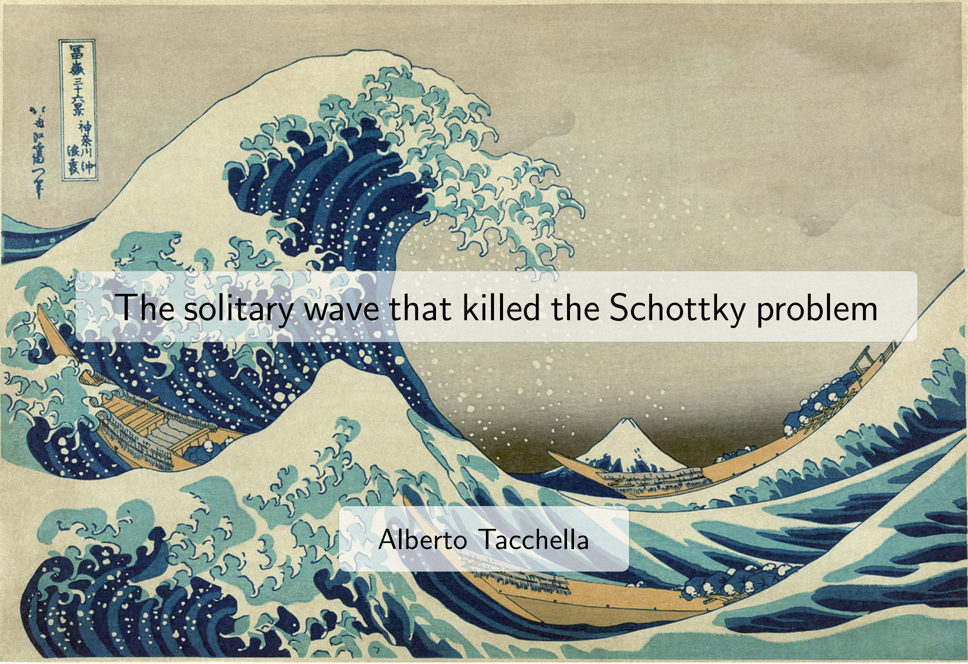
The solitary wave that killed the Schottky problem



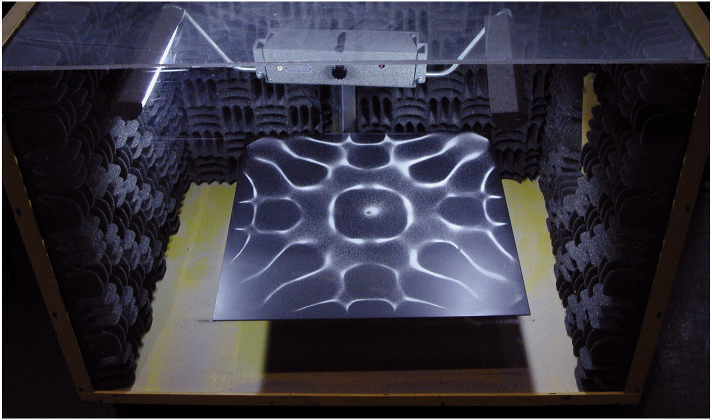
Alberto Tacchella

PhD seminars - April 27, 2016

Waves are everywhere...

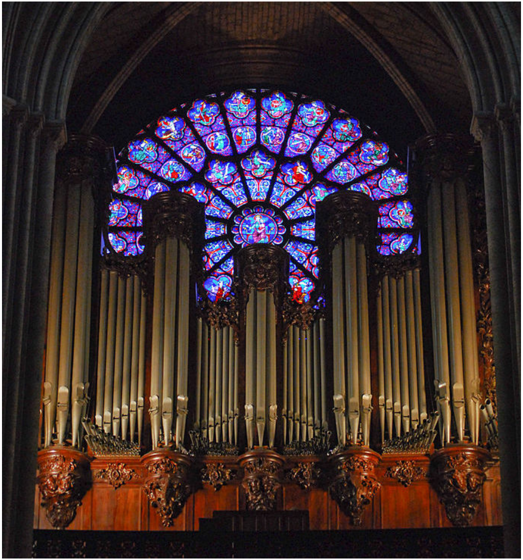
Surface waves at Ipanema, Rio de Janeiro, Brazil

Waves are everywhere...



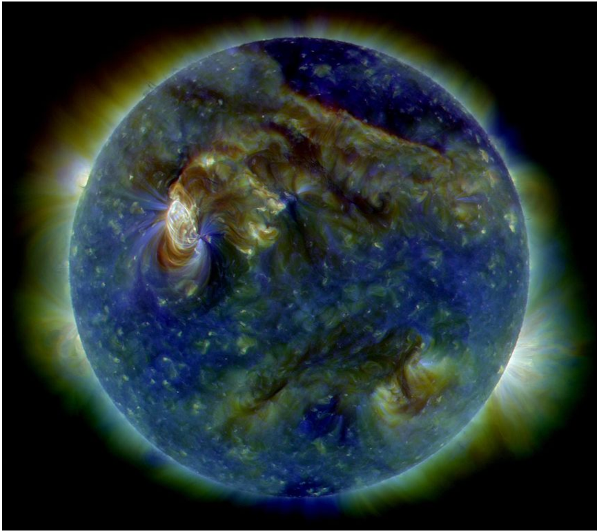
Chladni plate (Harvard Natural Sciences Lecture Demonstrations)

Waves are everywhere...



Pipe organ at Notre Dame Cathedral, Paris

Waves are everywhere...



Plasma waves in the sun (Solar Dynamics Observatory, NASA)

Waves are everywhere...

European Gravitational Observatory near Pisa, Italy

Linear waves

Wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u = 0$$

First derived (and solved) by d'Alembert (1747) in 1d case.
General solution:

$$u(x,t) = f(x-ct) + g(x+ct)$$

with f, g determined by initial data and/or boundary conditions.

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Examples of linear waves:

- electromagnetic (unless you are dealing with weird materials)
- acoustic (usually)

However, many other oscillatory phenomena (e.g. in fluids) are inherently nonlinear.

Nonlinear waves

Navier-Stokes equations for the velocity field $\vec{v}$:

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}\right)=-\vec{\nabla}p+\vec{\nabla}\cdot\mathbf{T}+\vec{f}$$

$+$ equation of state for the pressure

$+$ boundary conditions

ρ density, p pressure (unknown), $\mathbf{T}$ "deviatoric stress tensor" (depends on $\vec{v}$), $\vec{f}$ "body forces" acting on the fluid (gravity)

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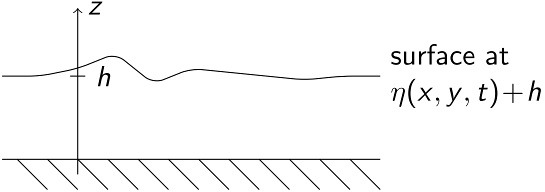
Highly nonlinear; existence and uniqueness of solutions with prescribed regularity is an open problem (worth $1M!)
Simplifications in the *physics* of the problem:

- *inviscid flow* ($\mathbf{T}=0$);
- *uncompressible flow* ($\vec{\nabla}\cdot\vec{v}=0$);
- *irrotational flow* ($\vec{\nabla}\times\vec{v}=0$); useful because we can rewrite our equations in terms of a potential φ such that $\vec{v}=\vec{\nabla}\varphi$.

Nonlinear waves

Simplifications in the *geometry* of the problem:



Nonlinear waves

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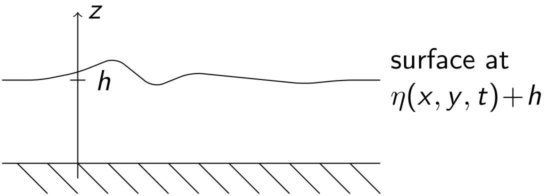

Making *all* these assumptions we are left with

$$\nabla^2 \varphi = 0$$

(continuity equation for flow in the bulk)

Nonlinear waves

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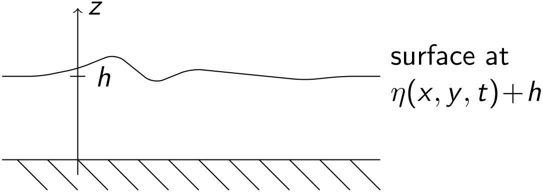


Making *all* these assumptions we are left with

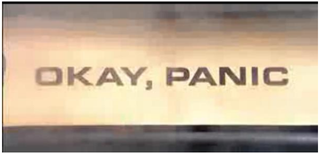
$$\nabla^2 \varphi = 0$$

(continuity equation for flow in the bulk) with boundary conditions:

$$\begin{cases} \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + g\eta = 0 & \text{on } z = \eta + h \\ \frac{\partial \varphi}{\partial z} = 0 & \text{on } z = 0 \\ \frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \varphi}{\partial z} = 0 & \text{on } z = \eta + h \end{cases}$$

Nonlinear waves, linearized

This system is still very difficult to solve, mainly because the boundary conditions involve the *unknown* function η.



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First fallback: *suppress dimensions!*

$$\frac{\partial \varphi}{\partial y} = 0 \qquad \frac{\partial \eta}{\partial y} = 0$$

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Second fallback: *linearize!*

$$\eta(x, t) = a \cos(k_0 x - \omega_0 t) \qquad a \ll h$$

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This ansatz satisfies our system when

$$\omega_0^2 = g k_0 \tanh(k_0 h)$$

dispersion relation $\Rightarrow$ phase velocity $c^2 = \frac{g}{k_0} \tanh(k_0 h)$.

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In the shallow water limit $k_0 h \to 0$,

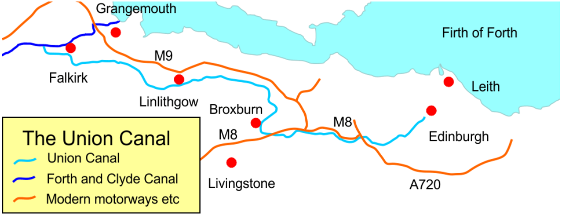
$$\omega_0^2 = g h k_0^2$$

No dispersion, waves with speed $c = \sqrt{gh}$

Solitary waves: discovery

John Scott Russell (1808–1882): scientist, civil engineer, naval architect, shipbuilder.

In 1834 he started performing experiments along the Union Canal, near Edimburgh, to find the most efficient design for canal boats.



One day he noticed a strange phenomenon...

Solitary waves: discovery

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped. Not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

Solitary waves: discovery

I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

"Report on waves", 1844

(and later in the report: *the Great Primary Wave of Translation*)

Solitary waves: discovery

Key characteristics of Russell's wave of translation:

- single hump (not periodic);
- stability in time (they do not flatten or steepen);
- speed $\sqrt{g(h+a)}$ (depends on the depth of the channel *and* the amplitude of the wave).

These features could not be explained using the theory of water waves available at the time.

Solitary waves: reception

G. B. Airy (1801–1892)

Airy 1845: *We are not disposed to recognize this wave as deserving the epithets "great" or "primary" [...] and we conceive that, ever since it was known that the theory of shallow waves of great length was contained in the [linear wave] equation, the theory of the solitary wave has been perfectly well known.*

Solitary waves: reception

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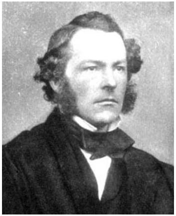

G. G. Stokes (1819–1903)

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Stokes 1846: *Thus the degradation in the height of such waves, which Mr. Russell observed, [...] is an essential characteristic of a solitary wave. It is true that this conclusion depends on an investigation which applies strictly to indefinitely small motions only: but [...] to disprove a general proposition it is sufficient to disprove a particular case.*

Solitary waves: explanations

Some years later the tide started turning...

J. V. Boussinesq
(1842–1929)

Experimental work is taken up in France by Darcy and Bazin; the results (published in 1865) confirm Russell's observations.
Motivated by these findings, Boussinesq develops in a 1872 paper what is known today as the "Boussinesq approximation" for nonlinear water waves, obtaining the PDE

$$\eta_{tt} - \eta_{xx} - (3\eta^2 + \eta_{xx})_{xx} = 0$$

which admits "solitary wave" solutions with the correct speed $c = \sqrt{g(h+a)}$.

Solitary waves: explanations

Some years later the tide started turning...

John Strutt, 3rd baron Rayleigh (1842–1919)

Independently, Rayleigh derives in 1876 another possible waveform for the solitary wave (but with no PDE):

is sufficiently defined by (I); but we may readily integrate again, so as to obtain the relation between x and y. Thus, if $l'-l=\beta, y-l=\eta$,

$$\pm x=\sqrt{\frac{l^2 l'}{3\beta}}\log_e\left\{\frac{2\beta}{\eta}-1+2\sqrt{\frac{\beta^2}{\eta^2}-\frac{\beta}{\eta}}\right\}, \quad . \quad (J)$$

the constant being taken so that $x=0$ when $\eta=\beta$. This equation gives the height η at any point x in terms of one constant, viz. the maximum height of the wave. There is therefore (in

Work on the topic continues in later years thanks to the Scottish mathematician John McCowan ("On the solitary wave", 1891).

A PhD problem

The solitary wave also captured the interest of Diederik Korteweg (1848–1941), professor at the University of Amsterdam in the Netherlands. He proposed this problem as a research topic to his student Gustav de Vries (1866–1934), who started a Ph.D. under his guidance in 1891.

Not everything went smoothly, however...

A PhD problem

From a letter sent by Korteweg to de Vries in October 1893:

To my regret I am unable to accept your dissertation in its present form. It contains too much translated material, where you follow Rayleigh and McCowan to the letter. The remarks and clarifications that you introduce now and then do not compensate for this shortcoming. The study of the literature concerning your subject matter must serve solely as a means for arriving at a more independent treatment, expressed in your own words and in accordance with your own line of reasoning, prompted, possibly, by the literature, which should not be followed so literally...

A PhD problem

From a letter sent by Korteweg to de Vries in October 1893:

It is obviously a disappointment for you who must have deemed to have already almost completed your task, to discover that you have apparently only completed the preparatory work. In the meantime do not be down-hearted. With pleasure I will do my best to help you mount the horse...

More than one year later (1 December 1894) de Vries successfully defended his thesis. The new results were written up in the paper "On the Change of Form of Long Waves advancing in a Rectangular Canal and on a New Type of Long Stationary Waves" published in the Philosophical Magazine in 1895.

The KdV equation

Their main result is the derivation of the famous KdV equation:

$$\eta_t + \frac{3}{2}\eta\eta_x + \frac{1}{4}\eta_{xxx} = 0$$

Two families of (explicit, exact) solutions:

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$$\eta_c(x,t) = \frac{2c}{\cosh^2(\sqrt{c}(x-ct))}$$

with speed c and height $2c$;

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$$\eta_c(x,t) = \frac{2c}{\cosh^2(\sqrt{c}(x-ct))}$$

with speed c and height $2c$; *cnoidal waves*

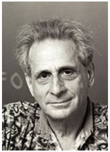
$$\eta_{a,m}(x,t) = a\,\mathrm{cn}^2(\frac{x-ct}{\Delta}, m)$$

where Δ and c can be expressed in terms of the wave height a and the *elliptic parameter* $0 < m < 1$. Two possible limits:

- for $m \to 1$, $\mathrm{cn}(z,m) \to 1/\cosh z$;
- for $m \to 0$, $\mathrm{cn}(z,m) \to \cos z$.

The birth of solitons

The KdV paper faded into obscurity for ~ 70 years. It was revived in the mid 60s thanks to two important developments:



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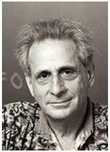

In 1965 Martin Kruskal (1925–2006) and Norman Zabusky (1929–) numerically studied the time evolution of the KdV equation for various initial conditions. Two discoveries:

- There are solutions which combine two solitary waves with different heights. When the slower wave starts in front, the evolution shows the faster wave reaching the slower one and (after a brief transient period) emerging in front, apparently undisturbed by the "interaction".

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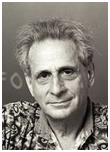


In 1965 Martin Kruskal (1925–2006) and Norman Zabusky (1929–) numerically studied the time evolution of the KdV equation for various initial conditions. Two discoveries:

- Any localized initial data evolves to a combination of a finite number of humps, each behaving like a solitary wave, plus some amount of "radiation" escaping in the opposite direction.

They propose the term soliton to describe these localized humps behaving somewhat like particles.

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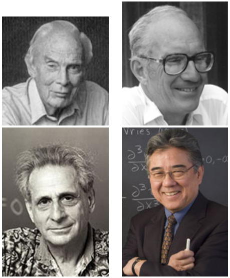

In a series of papers published during the years 1967–1969, Clifford Gardner (1924–2013), John Greene (1928–2007), M. Kruskal and Robert Miura (1938–) developed a new technique ("inverse scattering transform") to obtain exact solutions of the KdV equation. Among these are the "multisoliton solutions" found numerically by Kruskal and Zabusky.

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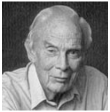
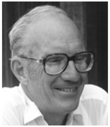
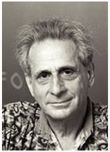
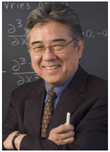

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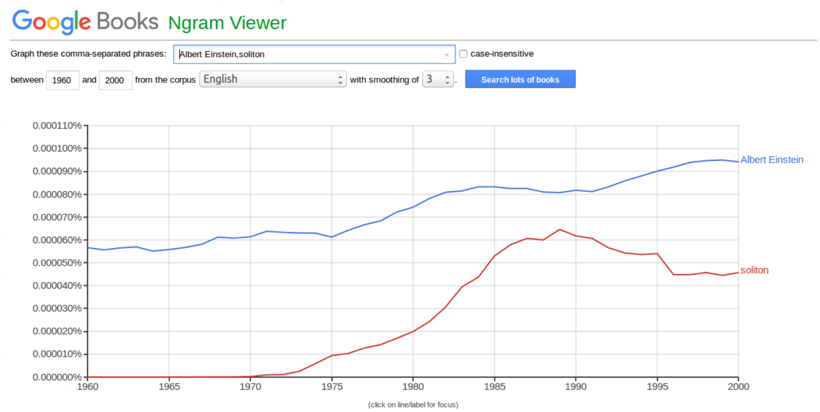
Existence of these solutions turns out to be related to a subtle property of the KdV equation: having an infinite number of constants of the motion (= functions of η and its derivatives that do not depend on t). Today we express this fact by saying that the KdV equation is completely integrable. Many other nonlinear PDEs (including the one derived by Boussinesq) share this property.

Solitons: the explosion

The following decades saw an explosive growth of works (papers, books, conferences...) related to soliton equations



Solitons: the explosion

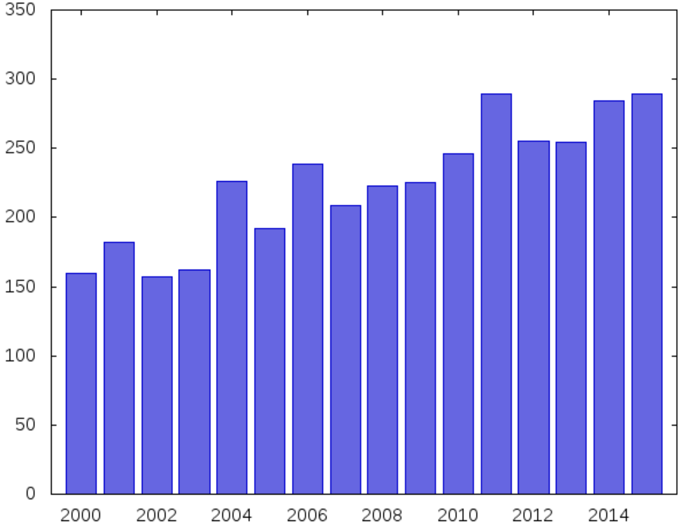
2010 AMS (sub)classes devoted to soliton equations

35Qxx	**Equations of mathematical physics and other areas of application [See also 35J05, 35J10, 35K05, 35L05]**
35Q05	Euler-Poisson-Darboux equations
35Q15	Riemann-Hilbert problems [See also 30E25, 31A25, 31B20]
35Q20	Boltzmann equations
35Q30	Navier-Stokes equations [See also 76D05, 76D07, 76N10]
35Q31	Euler equations [See also 76D05, 76D07, 76N10]
35Q35	PDEs in connection with fluid mechanics
35Q40	PDEs in connection with quantum mechanics
35Q41	Time-dependent Schrödinger equations, Dirac equations
35Q51	Soliton-like equations [See also 37K40]
35Q53	KdV-like equations (Korteweg-de Vries) [See also 37K10]
35Q55	NLS-like equations (nonlinear Schrödinger) [See also 37K10]
35Q56	Ginzburg-Landau equations
35Q60	PDEs in connection with optics and electromagnetic theory
35Q61	Maxwell equations

37Kxx	**Infinite-dimensional Hamiltonian systems [See also 35Axx, 35Qxx]**
37K05	Hamiltonian structures, symmetries, variational principles, conservation laws
37K10	Completely integrable systems, integrability tests, bi-Hamiltonian structures, hierarchies (KdV, KP, Toda, etc.)
37K15	Integration of completely integrable systems by inverse spectral and scattering methods
37K20	Relations with algebraic geometry, complex analysis, special functions [See also 14H70]
37K25	Relations with differential geometry
37K30	Relations with infinite-dimensional Lie algebras and other algebraic structures
37K35	Lie-Bäcklund and other transformations
37K40	Soliton theory, asymptotic behavior of solutions
37K45	Stability problems

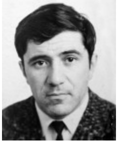
Solitons: the explosion

Number of papers on the arXiv with the word “soliton” in the title



The KP equation

Boris B. Kadomtsev (1928–1998)



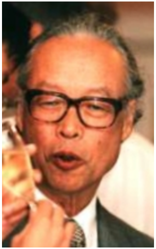
Vladimir I. Petviashvili (1936–1993)

An example of a soliton model in 2+1 dimensions is given by the KP equation, introduced in 1970 to describe the propagation of acoustic waves in plasma (related to research on fusion reactors):

$$\frac{3}{4}u_{yy} + \left(u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}\right)_x = 0$$

It is again derived in the hypothesis of essentially one-dimensional waves, but now allowing for a weak dispersive effect in the transverse direction (given by the u_{yy} term).
This equation is also completely integrable; it has many exact solutions, including rational functions, (multi)solitons, periodic and quasi-periodic ones.

Hirota's version of the KP equation



Ryogo Hirota
(1932–2015)

In 1971 Ryogo Hirota discovered a new way ("direct method") to generate soliton solutions for a large class of integrable equations. The key step is the introduction of a new dependent variable, the tau function, related to the original unknown function u by the nonlinear transformation

$$u = 2\frac{\partial^2}{\partial x^2}\log\tau$$

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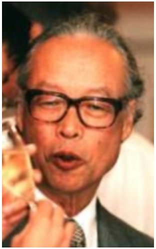

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Many soliton equations turn out to be equivalent to a new PDE which is bilinear in τ and its derivatives. For instance the KP equation in Hirota form reads

$$-3\tau_y^2 + 3\tau_{xx}^2 + 3\tau\tau_{yy} + 4\tau_t\tau_x - 4\tau\tau_{xt} - 4\tau_x\tau_{xxx} + \tau\tau_{xxxx} = 0$$

Apparently more complicated, but has a lot of hidden structure...

The geometry of the KP equation

Mikio Sato (1928–)

In 1981 Mikio Sato was able to give a rather explicit description of the space of all (formal) solutions to the KP equation as a certain infinite-dimensional Grassmannian.
A bit later (1986) Graeme Segal and George Wilson gave a more analytic interpretation of Sato's Grassmannian in terms of a certain kind of closed linear subspaces in a (complex, separable) Hilbert space.

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In 1981 Mikio Sato was able to give a rather explicit description of the space of all (formal) solutions to the KP equation as a certain infinite-dimensional Grassmannian.
A bit later (1986) Graeme Segal and George Wilson gave a more analytic interpretation of Sato's Grassmannian in terms of a certain kind of closed linear subspaces in a (complex, separable) Hilbert space.

To each point in the SSW Grassmannian we can associate a commutative algebra over $\mathbb{C}$ which is explicitly realized in terms of differential operators. In general the spectrum of this algebra will be 0-dimensional, but in some cases ("algebraic solutions") it is 1-dimensional…
$\Rightarrow$ algebraic curves enter the picture.

Essentials on algebraic curves (over $\mathbb{C}$)

Field extensions of $\mathbb{C}$ with tr. degree 1 $\longleftrightarrow$ Smooth irred. projective algebraic curves over $\mathbb{C}$

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Compact connected 1-dim. complex manifolds

Classified by a discrete invariant, the genus (number of holes), and a number of continuous moduli. Classically one introduces a *moduli space* $\mathfrak{M}_g$ for smooth irreducible algebraic curves of a fixed genus g. When $g > 1$, $\dim \mathfrak{M}_g = 3g - 3$.

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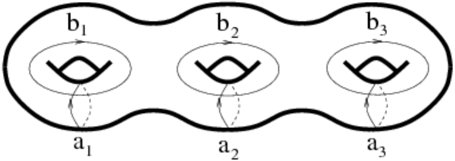
Fix a curve C of genus g; then $H_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$, isomorphism given by the choice of a standard basis



The Jacobian of an algebraic curve

Let K_C be the sheaf of regular/holomorphic differentials on C, $V := H^0(C, K_C)$; then $\dim V = g$. The discrete group $H_1(C, \mathbb{Z})$ embeds into V^* via the map sending a cycle γ to

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The image of this map defines a non-degenerate lattice Λ in $V^* \simeq \mathbb{C}^g$. It follows that the quotient

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is a g-dimensional complex torus, which is called the Jacobian variety of the curve C. We also have the *Abel-Jacobi map* $\phi \colon C \to J(C)$ which sends a point $p \in C$ to

$$\omega \mapsto \int_{p_0}^{p} \omega \mod \Lambda$$

Jacobians and abelian varieties

In fact we have more than just a complex torus: on V we have a positive definite Hermitian form given by

$$(\omega_1, \omega_2) = \int_C \omega_1 \wedge \overline{\omega_2}$$

which induces one on V^*, call it B. Also, on $H_1(C, \mathbb{Z}) \simeq \Lambda$ there is the intersection pairing

$$E\colon H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \to \mathbb{Z}$$

and the two are related by

$$E(x_1, x_2) = \operatorname{Im} B(x_1, x_2) \qquad \forall x_1, x_2 \in \Lambda$$

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Theorem (Riemann)

A complex torus $\mathbb{C}^g/\Lambda$ can be embedded in complex projective space iff there exists a positive definite Hermitian form B on $\mathbb{C}^g$ whose imaginary part takes integral values on $\Lambda \times \Lambda$.

Jacobians and abelian varieties

So $J(C)$ is really a complex torus (hence a group) which *also* happens to be a projective algebraic variety. Today these objects are called abelian varieties and are studied on any field; Riemann's theorem completely characterizes abelian varieties over $\mathbb{C}$.

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The Jacobian variety $J(C)$ comes also equipped with the Hermitian form B; in the general theory, this is called a polarization. Let $E := \operatorname{Im} B$; since it is skew-symmetric and integral on Λ, $\det E = (-1)^g d^2$ for some integer $d \geq 1$. A polarization with $d = 1$ is called principal.

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The polarization of $J(C)$ comes from the intersection form on $H_1(C, \mathbb{Z})$, which is unimodular, hence it is principal. We conclude that the Jacobian $J(C)$ is naturally a principally polarized abelian variety (p.p.a.v.).

The Schottky problem...

Summing up, the Jacobian construction gives a map

$$J\colon \mathfrak{M}_g \to \mathfrak{A}_g$$

where $\mathfrak{A}_g$ denotes the moduli space of p.p.a.v. of dimension g.

Theorem (Torelli)

The map J is injective (that is, one can reconstruct a curve from its Jacobian variety + polarization).

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Can J be also surjective? Not in general, as

$$\dim \mathfrak{A}_g = g(g+1)/2$$

and for $g > 3$ this is strictly bigger than $\dim \mathfrak{M}_g$. Then it is natural to consider the following problem: characterize those complex tori that arise as Jacobians of algebraic curves. This is the classical (Riemann-)Schottky problem.

...and its solution

Rather surprisingly, the Schottky problem has been solved in the 80s using the tools of soliton theory!

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To explain how, we need one last ingredient. Let $\mathbb{C}^g/\Lambda$ be an abelian variety with polarization B. Write $\Lambda = \Lambda_1 + \Lambda_2$ with $\Lambda_i \simeq \mathbb{Z}^g$ and $E(x_1, x_2) = 0$ when x_1, x_2 are both in Λ_1 or in Λ_2. For every $\alpha \in \Lambda_2$ define a (complex) linear map $\ell_\alpha \colon \mathbb{C}^g \to \mathbb{C}$ by

$$\ell_\alpha(x) = E(\alpha, x) \qquad \forall x \in \Lambda_1$$

Then the Riemann theta function associated to the chosen p.p.a.v. is the function $\theta \colon \mathbb{C}^g \to \mathbb{C}$ given by

$$\theta(x) = \sum_{\alpha \in \Lambda_2} \exp(2\pi \mathrm{i} \ell_\alpha(x) + \pi \mathrm{i} \ell_\alpha(\alpha))$$

(periodic in Λ_1 and quasi-periodic in Λ_2).

...and its solution

Rather surprisingly, the Schottky problem has been solved in the 80s using the tools of soliton theory!

In 1976 Igor Krichever proved that theta functions of Jacobians, when interpreted as tau functions, generate solutions of the KP equation. The converse was conjectured by Sergei Novikov and proved by Motohico Mulase (1984) and Takahiro Shiota (1986).

Theorem

An indecomposable p.p.a.v. X is the Jacobian of a curve of genus g if and only if there exist g-dimensional complex vectors $U \neq 0$, V and W such that the function

$$\tau(x, y, t) = \theta(Ux + Vy + Wt + Z)$$

is a solution of the KP equation in Hirota form for any $Z \in \mathbb{C}^g$.

Some sources

For the discovery of solitons and the KdV equation:

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For the Schottky problem:

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Thank you!