Integrable systems on quivers

Alberto Tacchella

Christmas Workshop on Quivers, Moduli Spaces
and Integrable Systems
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Outline of the talk

1. Associative geometry
2. Quiver path algebras
3. Abstract dynamical systems on quivers
4. Integrability of the induced systems

Geometry over an operad

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The first goal of this talk is to give a (very rough) idea of how associative geometry looks like in a special case. (Based on arXiv:1611.00644, to appear in JGP; part II hopefully available soon.)

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covariant objects

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$\pi \in \mathcal{V}^2(A)$ such that $[\pi, \pi]_S = 0$
$\Rightarrow$ *Poisson geometry*

Representation spaces

For every $d \in \mathbb{N}$ we have the affine scheme of d-dim. representations

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The group $\mathrm{GL}_d(\mathbb{K})$ acts on Rep_d^A by conjugation:

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Moduli space of representations (to be defined carefully[1]):

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Associative-geometric objects on A induce GL_d-invariant objects on Rep_d^A:

$$A_\natural \to \mathbb{K}[\mathrm{Rep}_d^A]^{\mathrm{GL}_d(\mathbb{K})}$$
$$\mathrm{DR}^p(A) \to \{\mathrm{GL}_d\text{-invariant } p\text{-forms on } \mathrm{Rep}_d^A\}$$
$$\mathcal{V}^p(A) \to \{\mathrm{GL}_d\text{-invariant } p\text{-vector fields on } \mathrm{Rep}_d^A\}$$

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Quivers

A quiver is a directed multigraph.

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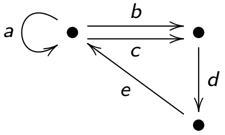
Random example:



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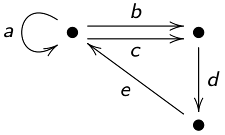

Each quiver Q determines an associative algebra, the path algebra $\mathbb{K}Q$.
Generated as a $\mathbb{K}$-vector space by paths (including the trivial ones), with product given by concatenation of paths

$$ba \qquad a^3 \qquad edba \qquad ca^2e \qquad ad = 0 \qquad \dots$$

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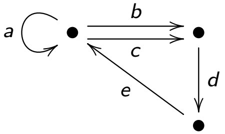

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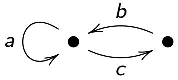
We can then apply the above machinery and do (associative) geometry over quivers. This is in fact a particularly nice case, as quiver path algebras are always "(formally) smooth".
How do geometric objects on $A = \mathbb{K}Q$ look like?

Fundamentals of associative geometry on $\mathbb{K}Q$

A regular function $f \in A_\natural$ is a sum of necklace words in A, that is cycles in the quiver Q:



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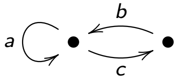

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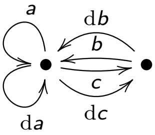
For each arrow $x \in Q$ add a parallel arrow $\mathrm{d}x$. An element $\alpha \in \Omega^p(A)$ is given by a path in this enlarged quiver with exactly p arrows of the form $\mathrm{d}x$:



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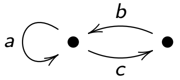

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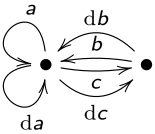

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Quotienting by $[A, \Omega^p(A)]$ the paths which are not closed become zero, so that for instance we have

$$bc\,\mathrm{d}b = [bc, \mathrm{d}b] = 0 \in \mathrm{DR}^1(A)$$
$$ca\,\mathrm{d}b = a\,\mathrm{d}b\,c = \mathrm{d}b\,ca \in \mathrm{DR}^1(A)$$
$$a^2\,\mathrm{d}b\,\mathrm{d}c = \mathrm{d}b\,\mathrm{d}c\,a^2 = -\mathrm{d}c\,a^2\,\mathrm{d}b \in \mathrm{DR}^2(A)$$

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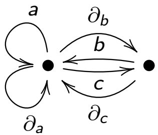
For each arrow $x \in Q$ add an opposite arrow ∂_x. An element $\theta \in \mathcal{D}^p(A)$ is given by a path in this enlarged quiver with exactly p arrows of the form ∂_x:



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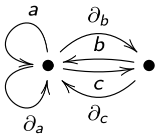

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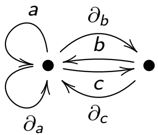

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The pairing between a 1-form $\alpha = \sum_{x\in Q} r_x \, \mathrm{d}x$ and a vector field $\theta = \sum_{x\in Q} p_x\partial_x$ is then given by

$$\langle \alpha, \theta \rangle = \sum_{x\in Q} r_x p_x \in A_\natural$$

Induced objects on representation spaces

$$(A, B, C) \in \mathrm{Rep}^{A}_{(n,r)} = \mathrm{Mat}_{n,n}(\mathbb{K}) \oplus \mathrm{Mat}_{n,r}(\mathbb{K}) \oplus \mathrm{Mat}_{r,n}(\mathbb{K})$$

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First principle: $\Omega^\bullet(A)$ and $\mathcal{D}^\bullet(A)$ induce "matrix-valued objects":

$p = bca \in A$	$\hat{p}(A, B, C) = BCA$
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Second principle: the passage to $\mathrm{DR}^\bullet(A)$ and $\mathcal{V}^\bullet(A)$ corresponds to "taking traces":

$$\begin{aligned}
p &= bca = cab = abc \in A_\natural & \hat{\hat{p}}(A,B,C) &= \mathrm{tr}\,BCA \in \mathbb{K}[\mathrm{Rep}^A_d]\\
\alpha &= bc\,\mathrm{d}b \in \mathrm{DR}^1(A) & \hat{\hat{\alpha}}(A,B,C) &= \mathrm{tr}\,BC\mathrm{d}B \in \Omega^1(\mathrm{Rep}^A_d)\\
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A dynamical system on a manifold M is (basically) a vector field on M. Translating in our setting, we get:

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A *dynamical system on a quiver* Q is an element of $\mathcal{V}^1(\mathbb{K}Q)$ (that is, a derivation $\mathbb{K}Q \to \mathbb{K}Q$).

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Every dynamical system θ on Q induces a family of GL_d-invariant global vector fields on representation spaces of $A = \mathbb{K}Q$:

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The flows of these vector fields are obtained by solving a system of matrix ODEs.

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$$\theta(x, y) = (\alpha + \beta yx, \gamma y^2 + \delta y)$$

$(\alpha, \beta, \gamma, \delta \in \mathbb{K})$ has been considered by Bruschi and Calogero (2006).

Hamiltonian systems on quivers

Things are easier if we have a symplectic or (more generally) a Poisson structure on A, in which case each regular function $H \in A_\natural$ automatically determines a corresponding "Hamiltonian derivation" θ_H given by

$$i_{\theta_H}(\omega) = -\mathrm{d}H \quad \text{resp.} \quad \pi(\mathrm{d}H, -)$$

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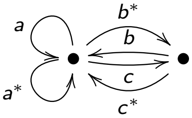
1. we have a large class of associative symplectic manifolds obtained by taking the double of a quiver:



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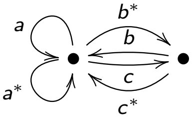

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2. we have a reduction process (Marsden-Weinstein quotient) which often gives "good" (smooth, symplectic) results.

(Unfortunately, associative symplectic forms are somewhat rare...)

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Many finite-dimensional Hamiltonian systems can be recovered in this way (rational/trigonometric/hyperbolic CM system, rational RS system, spin versions, external potentials, ...)

Liouville integrability of induced systems

Liouville-integrable system: Symplectic manifold (M, ω), $\dim M = 2n$, $(f_1, \dots, f_n)$ regular functions such that

1. $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$ almost everywhere on M;
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Example

Hamiltonian systems obtained by symplectic reduction of $T^*\mathfrak{g}$ with respect to the adjoint action of the Lie group G on $\mathfrak{g}$. In this case Q is the quiver with two loops ($=$ double of the Jordan quiver). These are generically Liouville-integrable.

Bihamiltonian integrability

A bihamiltonian manifold is a manifold M which admits two distinct Poisson bivectors π_0, π_1 such that $[\pi_0, \pi_1]_S = 0$. (Equivalently: $\pi_0 + \pi_1$ is Poisson, $\pi_0 + \lambda\pi_1$ is Poisson for every $\lambda \in \mathbb{P}^1$.)

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A bihamiltonian vector field

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The associative version of bihamiltonian manifolds is straightforward: triple (A, π_0, π_1) with $\pi_0, \pi_1 \in \mathcal{V}^2(A)$ double Poisson structures such that

$$[\pi_0, \pi_1]_S = 0 \in \mathcal{V}^3(A)$$

(which implies $[\hat{\tilde{\pi}}_0, \hat{\tilde{\pi}}_1]_S = 0$ on every representation space).

PN structures

Classically the following situation is quite typical: π_0 comes from a symplectic form, π_1 is obtained by means of a recursion operator $N\colon TM \to TM$ whose Nijenhuis torsion vanishes:

$$\mathcal{T}_N(X,Y) := [N(X),N(Y)] - N\big([N(X),Y] + [X,N(Y)] - N([X,Y])\big)$$

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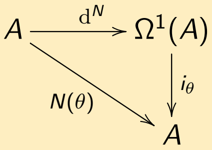
Is there a similar picture in the associative setting (at least for $A = \mathbb{K}Q$)?
Yes! (C. Bartocci, A.T., arXiv:1604.02012, to appear in LMP)

Definition

A linear map $N\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ is called *regular* if, for every $\theta \in \mathcal{V}^1(A)$, $N(\theta)$ factorizes as



for some derivation $\mathrm{d}^N\colon A \to \Omega^1(A)$.

PN structures

Definition

A *Nijenhuis tensor on A* is a regular map $N\colon \mathcal{V}^1(A) \to \mathcal{V}^1(A)$ such that $\mathcal{T}_N = 0$.

If the compatibility conditions

$$\begin{aligned} N \circ \tilde{\pi} = \tilde{\pi} \circ N^* &\quad (\text{as maps } \mathrm{DR}^1(A) \to \mathcal{V}^1(A)) \\ C_{(\pi,N)} = 0 &\quad (\text{Magri-Morosi concomitant}) \end{aligned}$$

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Theorem

Let Q be a quiver and (π, N) a Poisson-Nijenhuis structure on it. Then the bivector

$$\pi^N(\alpha, \beta) := \pi(N^*(\alpha), \beta) = \pi(\alpha, N^*(\beta))$$

is a double Poisson structure on A which is compatible with π.

Infinite-dimensional representations

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The most "economic" way of doing this is to take a direct limit:

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Theorem (Ginzburg)

The family of maps $\mathrm{tr}_d \colon A_\natural \to \mathbb{K}[\mathrm{Rep}^A_d]^{G_d}$ *(is compatible with the restrictions and) induces a bijection*

$$\mathrm{tr}_\infty \colon A_\natural \to \mathrm{prim}(\mathbb{K}[\mathrm{Rep}^A_\infty]^{G_\infty})$$

So the infinite-dimensional manifold Rep^A_∞ may actually be the "right" setting for associative geometry. However, it seems to be quite difficult to manage, even in simple cases.

Infinite-dimensional representations

Another idea: when Q is the Jordan quiver,

$$\mathsf{Rep}_d^{\mathbb{K}Q} = \mathsf{Mat}_{d,d}(\mathbb{K}) = \mathfrak{gl}_d(\mathbb{K})$$

Then: replace $\mathfrak{gl}_d(\mathbb{K})$ with a Kac-Moody algebra!
These have a Lie bracket and a trace map. Also, KM algebras can be studied from both the algebraic and the analytic side (loop spaces).

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These have a Lie bracket and a trace map. Also, KM algebras can be studied from both the algebraic and the analytic side (loop spaces).
This approach can already be found in the literature, as for instance in Nekrasov's approach to Calogero-Moser systems:

potential	real system	complexified system
rational	$T^*\mathfrak{su}_n$	$T^*\mathfrak{sl}_n(\mathbb{C})$
trigonometric	$T^*\widehat{\mathfrak{su}}_n$	$T^*\widehat{\mathfrak{sl}}_n(\mathbb{C})$
elliptic	?	$T^*\widehat{\mathfrak{sl}}_n(\mathbb{C})^{\Sigma}$