Calogero-Moser systems from associative geometry

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Outline of the talk

General idea

Natural (?) extension of the dualities between geometry & algebra we all know and love:

$$\mathbf{Spc} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathbf{Alg}^{\mathrm{op}}$$

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- Generalize Gelfand duality to not-necessarily-commutative C*-algebras (Connes);
- Concentrate on algebras where the failure of commutativity is somewhat "under control" (e.g. graded-commutative, UEA of Lie algebras, rings of differential operators);
- Develop a theory that works for generic associative algebras (e.g. free ones) ⇒ associative geometry.

A guiding principle: the "Kontsevich philosophy"

Let A be a finitely generated associative algebra over a field $\mathbb{K}$. For every $d \in \mathbb{N}$ we have the affine scheme

$$\mathrm{Rep}_d^A := \mathrm{Hom}_{\mathbb{K}\text{-}\mathbf{Alg}}(A, \mathrm{Mat}_{d,d}(\mathbb{K}))$$

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Baby version: associative-geometric objects act like “blueprints” for an infinite sequence of (commutative-)geometric objects.

Smoothness in associative geometry

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Regular functions

Transposing the evaluation map $\mathrm{Rep}_d^A \times A \to \mathrm{Mat}_{d,d}(\mathbb{C})$ we can build a map

$$\begin{array}{ccccc} \mathrm{Rep}_d^A & \to & \mathrm{Mat}_{d,d}(\mathbb{C}) & \to & \mathbb{C} \\ \rho & \mapsto & \rho(a) & \mapsto & \mathrm{tr}\,\rho(a) \end{array}$$

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$$\frac{A}{[A, A]} \to \mathbb{C}[\mathrm{Rep}_d^A]^{\mathrm{GL}_d(\mathbb{C})}$$

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Thus we can interpret $A/[A, A]$ as the (linear!) space of regular functions on the associative space determined by A.

Vector fields, differential forms

Every derivation $\theta\colon A \to A$ induces a $\mathrm{GL}_d(\mathbb{C})$-invariant vector field $\hat{\theta}$ on Rep_d^A. When $A = \mathbb{C}\langle x_1, \ldots, x_n\rangle$ and θ is defined by $x_i \mapsto f_i(x_1, \ldots, x_n)$, the corresponding vector field on Rep_d^A is

$$\hat{\theta}_{(X_1,\ldots,X_n)} = (f_1(X_1, \ldots, X_n), \ldots, f_n(X_1, \ldots, X_n))$$

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Kähler differentials: A-bimodule $\Omega^1(A) \simeq A \otimes \bar{A}$ (where $\bar{A} := A/\mathbb{C}$) equipped with a derivation $\mathrm{d}\colon A \to \Omega^1(A)$ defined by $a \mapsto 1 \otimes \bar{a}$. For example when A is free an element like $\alpha = f \otimes \bar{x}_1 \in \Omega^1(A)$ corresponds to the differential form on Rep_d^A

$$\hat\alpha_{(X_1,\ldots,X_n)} = \mathrm{tr}\, f(X_1, \ldots, X_n)\mathrm{d}X_1$$

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The map $\alpha \mapsto \hat{\alpha}$ vanishes on the linear subspace $[A, \Omega^1(A)]$, so it makes sense to consider instead

$$\mathrm{DR}^1(A) = \frac{\Omega^1(A)}{[A, \Omega^1(A)]}$$

as the linear space of differential forms on A.

p-forms

One can extend in a universal way the pair $(\Omega^1(A), \mathrm{d})$ to a dg-algebra $(\Omega^\bullet(A), \mathrm{d})$. Concretely, $\Omega^n(A) \simeq A \otimes \bar{A} \otimes \cdots \otimes \bar{A}$ so that an element $\omega \in \Omega^n(A)$ can be written as $a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n$.

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Let us define the Karoubi de Rham complex as

$$\mathrm{DR}^\bullet(A) := \frac{\Omega^\bullet(A)}{[\Omega^\bullet(A), \Omega^\bullet(A)]}$$

where $[a, b] = ab - (-1)^{|a||b|} ba$ is the *graded* commutator.

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where $[a, b] = ab - (-1)^{|a||b|} ba$ is the *graded* commutator. Associative p-forms live in $\mathrm{DR}^\bullet(A)$ and induce p-forms on Rep_d^A. In the free case, if we take for instance $\omega = f \otimes \bar{x}_2 \otimes \bar{x}_1 \in \Omega^2(A)$ then $[\omega] \in \mathrm{DR}^2(A)$ corresponds to

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In fact we can define a whole *Cartan calculus* on $\mathrm{DR}^\bullet(A)$, that is a degree -1 "interior product" i_θ and a degree 0 "Lie derivative" $\mathcal{L}_\theta$ such that $\mathcal{L}_\theta = [\mathrm{d}, i_\theta]$. This makes it possible to develop a *symplectic geometry* for associative spaces.

p-vectors

Suppose $A = \mathbb{C}Q$. Let $\overline{Q}$ be the *double* of Q, obtained by adding for each arrow x in Q an arrow ∂_x going in the opposite direction. We grade the path algebra $\mathbb{C}\overline{Q}$ by the number of ∂-arrows and define

$$\mathcal{V}^\bullet(A) := \frac{\mathbb{C}\overline{Q}}{[\mathbb{C}\overline{Q}, \mathbb{C}\overline{Q}]}$$

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For instance a path $\xi = f\partial_{x_2} g\partial_{x_1}$ in $\mathbb{C}\overline{Q}$ corresponds to

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We can also define a Schouten bracket

$$[\mathcal{V}^p(A), \mathcal{V}^q(A)] \subseteq \mathcal{V}^{p+q-1}(A)$$

corresponding to the usual one on Rep_d^A. This makes it possible to develop a *Poisson geometry* for associative spaces.

An example: the associative plane

Let us consider $A = \mathbb{C}\langle x, y\rangle$, the "associative affine plane". $\mathrm{Rep}_d^A = \mathrm{Mat}_{d,d}(\mathbb{C}) \oplus \mathrm{Mat}_{d,d}(\mathbb{C})$; its coordinate algebra is a polynomial ring in the $2d^2$ indeterminates X_{ij}, Y_{ij} $(i, j = 1 \ldots d)$.

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One also has bivectors like $\pi = h_1\partial_x\partial_x + h_2\partial_x h_3\partial_y \in \mathcal{V}^2(A)$ (not the most general one!), which corresponds to

$$\hat{\pi}_{(X,Y)}((A_1, B_1), (A_2, B_2)) = \operatorname{tr}(h_1 A_1 A_2 - A_1 h_1 A_2 + h_2 A_1 h_3 B_2 - h_3 B_1 h_2 A_2)$$

and so on...

Symplectic and Poisson geometry on the associative plane

We would like to do Hamiltonian mechanics on A. Natural starting point is to take $\omega = \mathrm{d}y\,\mathrm{d}x \in \mathsf{DR}^2(A)$, which gives

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canonical symplectic structures on each $\mathrm{Rep}_d^A \simeq T^* \mathrm{Mat}_{d,d}(\mathbb{C})$. Linear Poisson bivectors on A are in 1-1 correspondence with associative algebra structures on $\mathbb{C}^2$. There are exactly 6 of them:

$x\partial_x\partial_x$	Lie-Poisson on $X \oplus$ zero bracket on Y
$x\partial_x\partial_x + y\partial_y\partial_y$	Lie-Poisson on $X \oplus$ Lie-Poisson on Y
$y\partial_x\partial_x$	?
$x\partial_x\partial_x + y\partial_x\partial_y$	$\mathfrak{gl}_d(\mathbb{C}) \ltimes \mathrm{Mat}_{d,d}(\mathbb{C})$ by left mult.
$x\partial_x\partial_x + y\partial_y\partial_x$	$\mathfrak{gl}_d(\mathbb{C}) \ltimes \mathrm{Mat}_{d,d}(\mathbb{C})$ by right mult.
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Symplectic and Poisson geometry on the associative plane

We would like to do Hamiltonian mechanics on A. Natural starting point is to take $\omega = \mathrm{d}y\,\mathrm{d}x \in \mathrm{DR}^2(A)$, which gives

$$\hat{\omega}_{(X,Y)}((U_1, V_1), (U_2, V_2)) = \mathrm{tr}(V_1 U_2 - U_1 V_2)$$

canonical symplectic structures on each $\mathrm{Rep}_d^A \simeq T^* \mathrm{Mat}_{d,d}(\mathbb{C})$. Linear Poisson bivectors on A are in 1-1 correspondence with associative algebra structures on $\mathbb{C}^2$. There are exactly 6 of them:

$x\partial_x\partial_x$	Lie-Poisson on $X \oplus$ zero bracket on Y
$x\partial_x\partial_x + y\partial_y\partial_y$	Lie-Poisson on $X \oplus$ Lie-Poisson on Y
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(This correspondence holds in general, e.g. for associative 3d space one has 21 distinct structures + a single $\mathbb{P}^1$-indexed family.)

Symplectic and Poisson geometry on the associative plane

Quadratic Poisson structures on $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ have been classified (Odesskii, Rubtsov, Sokolov 2012). They are controlled by two tensors $r^{\gamma\delta}_{\alpha\beta}$ and $a^{\gamma\delta}_{\alpha\beta}$ (with indices running on $1 \ldots n$) satisfying a set of relations (when $a = 0$: associative YB equation for r).

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For $n = 2$ there are 7 quadratic Poisson structures:

$$
\begin{array}{ll}
y\partial_y x\partial_y & (a=0) \\
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y\partial_y y\partial_x & (a=0) \\
y\partial_y y\partial_x + y^2\partial_x\partial_y + xy\partial_x\partial_x & \\
y\partial_y y\partial_x - y^2\partial_y\partial_x - yx\partial_x\partial_x & \\
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Of course there are also Poisson bivectors of higher degrees (e.g. $x\partial_x x^2\partial_x$, $x\partial_x x^2\partial_x + x\partial_x xy\partial_y$, $\dots$).

Applications: Calogero-Moser systems

Work in the "associative symplectic manifold" (A, ω), $\omega = \mathrm{d}y\,\mathrm{d}x$. *Free motion* on A is described by $H = \frac{1}{2}y^2$. Hamilton equations are $\dot{x} = y$, $\dot{y} = 0$. Their general solution is

$$x(t) = y_0 t + x_0 \qquad (x_0, y_0 \in A)$$

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This flow is not terribly interesting on $\mathrm{Rep}_d^A = T^* \mathrm{Mat}_{d,d}(\mathbb{C})$, but we can perform a symplectic quotient using the $\mathrm{GL}_d(\mathbb{C})$-action:

$$\mu\colon \mathrm{Rep}_d^A \to \mathfrak{gl}_d(\mathbb{C}) \qquad \mu(X, Y) = [X, Y]$$

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is a smooth symplectic manifold of dimension $2d$. The open dense subset where X is diagonalizable is symplectomorphic to $T^*\mathbb{C}^{(d)}$ and the above flow corresponds to the flow determined by

$$H = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i \neq j}(q_i - q_j)^{-2}$$

Applications: Calogero-Moser systems

Still working on (A, ω), let us consider the "harmonic oscillator" $H = \frac{1}{2}(y^2 + \omega^2 x^2)$. Hamilton equations are $\dot{x} = y$, $\dot{y} = -\omega^2 x$. Their general solution is

$$x(t) = x_0 \cos(\omega t) + \frac{y_0}{\omega} \sin(\omega t)$$

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We descend again to the symplectic quotient $\mathcal{C}_d$ (defined as before). On the dense open subset where X is diagonalizable the above flow corresponds to the flow determined by

$$H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \left((q_i - q_j)^{-2} + \omega^2 (q_i - q_j)^2 \right)$$

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One can also consider bigger adjoint orbits in $\mathfrak{gl}_d(\mathbb{C})$ (with some complications) $\Rightarrow$ Calogero-Moser systems with "spin"

$$H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\lambda_{ij}^2}{(q_i - q_j)^2}$$

Applications: Calogero-Moser systems

Consider now the "associative Poisson manifold" (A, π) where $\pi = y\partial_y\partial_y + x\partial_y\partial_x$. On the open subset of Rep_d^A where X is invertible the Poisson bivector $\hat{\pi}$ is non degenerate. The corresponding symplectic form reads

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We again perform a symplectic quotient with respect to the $\mathrm{GL}_d(\mathbb{C})$-action, but this time with momentum map

$$\mu \colon \mathcal{U} \to \mathfrak{gl}_d(\mathbb{C}) \qquad \mu(X, Y) = XYX^{-1} - Y$$

We obtain another smooth symplectic manifold of dimension $2d$

$$\mathcal{C}_d^{tr} := \mu^{-1}(\mathbb{O}_1)/\,\mathrm{GL}_d(\mathbb{C})$$

Applications: Calogero-Moser systems

Let us take again $H = \frac{1}{2}y^2$, with solution $x(t) = y_0 t + x_0$. The projection of this flow on the open dense subset of $\mathcal{C}_d^{tr}$ where X is diagonalizable corresponds to the flow determined by

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Many other systems may be amenable to such a treatment:

- Gibbons-Hermsen
- CM systems associated to root systems of Lie algebras $\neq A_n$
- non-periodic Toda lattice
- every other classical integrable system mentioned in Perelomov's book

Thank you very much for your attention and...

Happy birthday Ugo!