Noncommutative symplectic geometry of Gibbons-Hermsen varieties

Alberto Tacchella
ICMC - USP São Carlos
tacchella@icmc.usp.br

EGBI 2012 - Salvador
July 30, 2012

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Some consequences:

- isomorphism $\tilde{\omega}\colon \mathcal{X}(M) \to \Omega^1(M)$ defined by $X \mapsto i_X(\omega)$;

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Some consequences:

- isomorphism $\tilde{\omega}\colon \mathcal{X}(M) \to \Omega^1(M)$ defined by $X \mapsto i_X(\omega)$;
- map $X\colon \Omega^0(M) \to \mathcal{X}(M)$ defined by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Some consequences:

- isomorphism $\tilde{\omega}\colon \mathcal{X}(M) \to \Omega^1(M)$ defined by $X \mapsto i_X(\omega)$;
- map $X\colon \Omega^0(M) \to \mathcal{X}(M)$ defined by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$
- ...which gives a *symplectic* vector field (i.e. $\mathcal{L}_{X_f}(\omega) = 0$);

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Some consequences:

- isomorphism $\tilde{\omega}\colon \mathcal{X}(M) \to \Omega^1(M)$ defined by $X \mapsto i_X(\omega)$;
- map $X\colon \Omega^0(M) \to \mathcal{X}(M)$ defined by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$
- ...which gives a *symplectic* vector field (i.e. $\mathcal{L}_{X_f}(\omega) = 0$);
- chain of inclusions: $\mathcal{X}(M) \supseteq \mathcal{X}_{\mathrm{symp}}(M) \supseteq \mathcal{X}_{\mathrm{Ham}}(M)$

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Some consequences:

- isomorphism $\tilde{\omega}\colon \mathcal{X}(M) \to \Omega^1(M)$ defined by $X \mapsto i_X(\omega)$;
- map $X\colon \Omega^0(M) \to \mathcal{X}(M)$ defined by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$
- ...which gives a *symplectic* vector field (i.e. $\mathcal{L}_{X_f}(\omega) = 0$);
- chain of inclusions: $\mathcal{X}(M) \supseteq \mathcal{X}_{\mathrm{symp}}(M) \supseteq \mathcal{X}_{\mathrm{Ham}}(M)$
- ...corresponding to $\Omega^1(M) \supseteq \Omega^1_{\mathrm{closed}}(M) \supseteq \Omega^1_{\mathrm{exact}}(M)$;

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Some consequences:

- isomorphism $\tilde{\omega}\colon \mathcal{X}(M) \to \Omega^1(M)$ defined by $X \mapsto i_X(\omega)$;
- map $X\colon \Omega^0(M) \to \mathcal{X}(M)$ defined by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$
- ...which gives a *symplectic* vector field (i.e. $\mathcal{L}_{X_f}(\omega) = 0$);
- chain of inclusions: $\mathcal{X}(M) \supseteq \mathcal{X}_{\mathrm{symp}}(M) \supseteq \mathcal{X}_{\mathrm{Ham}}(M)$
- ...corresponding to $\Omega^1(M) \supseteq \Omega^1_{\mathrm{closed}}(M) \supseteq \Omega^1_{\mathrm{exact}}(M)$;
- *Poisson brackets*, defined e.g. by $\{f, g\} := \omega(X_f, X_g)$;

Symplectic geometry in one slide

Basic ingredients:

- a (smooth) variety M of even dimension;
- a closed, non-degenerate 2-form ω on M.

Some consequences:

- isomorphism $\tilde{\omega}\colon \mathcal{X}(M) \to \Omega^1(M)$ defined by $X \mapsto i_X(\omega)$;
- map $X\colon \Omega^0(M) \to \mathcal{X}(M)$ defined by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$
- ...which gives a *symplectic* vector field (i.e. $\mathcal{L}_{X_f}(\omega) = 0$);
- chain of inclusions: $\mathcal{X}(M) \supseteq \mathcal{X}_{\mathrm{symp}}(M) \supseteq \mathcal{X}_{\mathrm{Ham}}(M)$
- ...corresponding to $\Omega^1(M) \supseteq \Omega^1_{\mathrm{closed}}(M) \supseteq \Omega^1_{\mathrm{exact}}(M)$;
- *Poisson brackets*, defined e.g. by $\{f, g\} := \omega(X_f, X_g)$;
- an exact sequence of Lie algebras $0 \to H^0_{\mathrm{dR}}(M) \to \Omega^0(M) \to \mathcal{X}_{\mathrm{symp}}(M) \to H^1_{\mathrm{dR}}(M) \to 0.$

Non-commutative geometry (à la Ginzburg)

Founding principle of non-commutative geometry: generalize the well-known dualities of the form

$$\mathbf{Spc} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathbf{Alg}^{\mathrm{op}}$$

to the case where **Alg** is some category of non-commutative algebras.

Non-commutative geometry (à la Ginzburg)

Founding principle of non-commutative geometry: generalize the well-known dualities of the form

$$\mathbf{Spc} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathbf{Alg}^{\mathrm{op}}$$

to the case where **Alg** is some category of non-commutative algebras.

Ginzburg's approach: for A any associative algebra,

- take the familiar notions of (commutative) algebraic geometry and replace A-modules with A-bimodules;

Non-commutative geometry (à la Ginzburg)

Founding principle of non-commutative geometry: generalize the well-known dualities of the form

$$\mathbf{Spc} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathbf{Alg}^{\mathrm{op}}$$

to the case where **Alg** is some category of non-commutative algebras.

Ginzburg's approach: for A any associative algebra,

- take the familiar notions of (commutative) algebraic geometry and replace A-modules with A-bimodules;
- do not focus on particular constructions (which are not going to work anymore), but on universal properties;

Non-commutative geometry (à la Ginzburg)

Founding principle of non-commutative geometry: generalize the well-known dualities of the form

$$\mathbf{Spc} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathbf{Alg}^{\mathrm{op}}$$

to the case where **Alg** is some category of non-commutative algebras.

Ginzburg's approach: for A any associative algebra,

- take the familiar notions of (commutative) algebraic geometry and replace A-modules with A-bimodules;
- do not focus on particular constructions (which are not going to work anymore), but on universal properties;
- might get something different even when A is commutative!

Some constructions that generalize

- Vector fields as A-bimodule derivations $A \to A$;

Some constructions that generalize

- Vector fields as A-bimodule derivations $A \to A$;
- Kähler differentials as a universal element $(\Omega^1_{\mathrm{nc}}, \mathrm{d})$ for the functor $\mathrm{Der}(A, -)\colon A\text{-}\mathbf{Bimod} \to \mathbf{Set}$;

Some constructions that generalize

- Vector fields as A-bimodule derivations $A \to A$;
- Kähler differentials as a universal element $(\Omega^1_{\mathrm{nc}}, \mathrm{d})$ for the functor $\mathrm{Der}(A, -)\colon A\text{-}\mathbf{Bimod} \to \mathbf{Set}$;
- Noncommutative forms $\Omega^\bullet_{\mathrm{nc}}(A)$ as elements of the tensor algebra of the A-bimodule $\Omega^1_{\mathrm{nc}}(A)$;

Some constructions that generalize

- Vector fields as *A*-bimodule derivations $A \to A$;
- Kähler differentials as a universal element $(\Omega^1_{\mathrm{nc}}, \mathrm{d})$ for the functor $\mathrm{Der}(A, -)\colon A\text{-}\mathbf{Bimod} \to \mathbf{Set}$;
- Noncommutative forms $\Omega^\bullet_{\mathrm{nc}}(A)$ as elements of the tensor algebra of the *A*-bimodule $\Omega^1_{\mathrm{nc}}(A)$;
- Cartan calculus: a degree -1 "interior product" i_θ and a degree 0 "Lie derivative" $\mathcal{L}_\theta$ satisfying

$$\mathcal{L}_\theta = [\mathrm{d}, i_\theta]$$

$$[\mathcal{L}_\theta, \mathcal{L}_\eta] = \mathcal{L}_{[\theta,\eta]}$$

$$[\mathcal{L}_\theta, i_\gamma] = i_{[\theta,\gamma]}$$

The complex of non-commutative differential forms

Hence we have a complex $(\Omega^\bullet_{\mathrm{nc}}(A), \mathrm{d})$. However

$$H^k(\Omega^\bullet_{\mathrm{nc}}(A)) = \begin{cases} \mathbb{K} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

The complex of non-commutative differential forms

Hence we have a complex $(\Omega^\bullet_{\mathrm{nc}}(A), \mathrm{d})$. However

$$H^k(\Omega^\bullet_{\mathrm{nc}}(A)) = \begin{cases} \mathbb{K} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

More interesting is the Karoubi-de Rham complex, defined by

$$\mathrm{DR}^\bullet(A) := \frac{\Omega^\bullet_{\mathrm{nc}}(A)}{[\Omega^\bullet_{\mathrm{nc}}(A), \Omega^\bullet_{\mathrm{nc}}(A)]}$$

as a quotient of graded vector spaces, with

$$[a, b] = ab - (-1)^{\deg a \deg b} ba$$

The complex of non-commutative differential forms

Hence we have a complex $(\Omega^\bullet_{\mathrm{nc}}(A), \mathrm{d})$. However

$$H^k(\Omega^\bullet_{\mathrm{nc}}(A)) = \begin{cases} \mathbb{K} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

More interesting is the Karoubi-de Rham complex, defined by

$$\mathrm{DR}^\bullet(A) := \frac{\Omega^\bullet_{\mathrm{nc}}(A)}{[\Omega^\bullet_{\mathrm{nc}}(A), \Omega^\bullet_{\mathrm{nc}}(A)]}$$

as a quotient of graded vector spaces, with

$$[a, b] = ab - (-1)^{\deg a \deg b} ba$$

d, i, $\mathcal{L}$ all descend on $\mathrm{DR}^\bullet(A)$.

The complex of non-commutative differential forms

Hence we have a complex $(\Omega^\bullet_{\mathrm{nc}}(A), \mathrm{d})$. However

$$H^k(\Omega^\bullet_{\mathrm{nc}}(A)) = \begin{cases} \mathbb{K} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

More interesting is the Karoubi-de Rham complex, defined by

$$\mathrm{DR}^\bullet(A) := \frac{\Omega^\bullet_{\mathrm{nc}}(A)}{[\Omega^\bullet_{\mathrm{nc}}(A), \Omega^\bullet_{\mathrm{nc}}(A)]}$$

as a quotient of graded vector spaces, with

$$[a, b] = ab - (-1)^{\deg a \deg b} ba$$

d, i, $\mathcal{L}$ all descend on $\mathrm{DR}^\bullet(A)$.

$$\mathrm{DR}^0(A) = \frac{A}{[A, A]}$$

Non-commutative symplectic geometry

A non-commutative symplectic manifold is given by an associative algebra A and a closed, non-degenerate 2-form $\omega \in \mathrm{DR}^2(A)$.

Non-commutative symplectic geometry

A non-commutative symplectic manifold is given by an associative algebra A and a closed, non-degenerate 2-form $\omega \in \mathrm{DR}^2(A)$.

- isomorphism $\tilde{\omega}\colon \mathrm{Der}(A) \to \mathrm{DR}^1(A)$ defined by $\theta \mapsto i_\theta(\omega)$;

Non-commutative symplectic geometry

A non-commutative symplectic manifold is given by an associative algebra A and a closed, non-degenerate 2-form $\omega \in \mathrm{DR}^2(A)$.

- isomorphism $\tilde{\omega}\colon \mathrm{Der}(A) \to \mathrm{DR}^1(A)$ defined by $\theta \mapsto i_\theta(\omega)$;
- symplectic derivations ($\mathcal{L}_\theta(\omega) = 0$) forming a Lie subalgebra $\mathrm{Der}_\omega(A) \subseteq \mathrm{Der}(A)$;

Non-commutative symplectic geometry

A non-commutative symplectic manifold is given by an associative algebra A and a closed, non-degenerate 2-form $\omega \in \mathrm{DR}^2(A)$.

- isomorphism $\tilde{\omega}\colon \mathrm{Der}(A) \to \mathrm{DR}^1(A)$ defined by $\theta \mapsto i_\theta(\omega)$;
- symplectic derivations ($\mathcal{L}_\theta(\omega) = 0$) forming a Lie subalgebra $\mathrm{Der}_\omega(A) \subseteq \mathrm{Der}(A)$;
- map $\theta\colon \mathrm{DR}^0(A) \to \mathrm{Der}_\omega(A)$ given by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$ ("Hamiltonian derivation" associated to f);

Non-commutative symplectic geometry

A non-commutative symplectic manifold is given by an associative algebra A and a closed, non-degenerate 2-form $\omega \in \mathrm{DR}^2(A)$.

- isomorphism $\tilde{\omega}\colon\ \mathrm{Der}(A) \to \mathrm{DR}^1(A)$ defined by $\theta \mapsto i_\theta(\omega)$;
- symplectic derivations ($\mathcal{L}_\theta(\omega) = 0$) forming a Lie subalgebra $\mathrm{Der}_\omega(A) \subseteq \mathrm{Der}(A)$;
- map $\theta\colon\ \mathrm{DR}^0(A) \to \mathrm{Der}_\omega(A)$ given by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$ ("Hamiltonian derivation" associated to f);
- θ is symplectic iff $i_\theta(\omega)$ is closed in $\mathrm{DR}^1(A)$, hamiltonian iff $i_\theta(\omega)$ is exact in $\mathrm{DR}^1(A)$;

Non-commutative symplectic geometry

A non-commutative symplectic manifold is given by an associative algebra A and a closed, non-degenerate 2-form $\omega \in \mathrm{DR}^2(A)$.

- isomorphism $\tilde{\omega}\colon \mathrm{Der}(A) \to \mathrm{DR}^1(A)$ defined by $\theta \mapsto i_\theta(\omega)$;
- symplectic derivations ($\mathcal{L}_\theta(\omega) = 0$) forming a Lie subalgebra $\mathrm{Der}_\omega(A) \subseteq \mathrm{Der}(A)$;
- map $\theta\colon \mathrm{DR}^0(A) \to \mathrm{Der}_\omega(A)$ given by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$ ("Hamiltonian derivation" associated to f);
- θ is symplectic iff $i_\theta(\omega)$ is closed in $\mathrm{DR}^1(A)$, hamiltonian iff $i_\theta(\omega)$ is exact in $\mathrm{DR}^1(A)$;
- Poisson brackets $\{f, g\} := i_{\theta_f} i_{\theta_g}(\omega)$ that make $\mathrm{DR}^0(A)$ a Lie algebra;

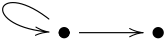
Non-commutative symplectic geometry

A non-commutative symplectic manifold is given by an associative algebra A and a closed, non-degenerate 2-form $\omega \in \mathrm{DR}^2(A)$.

- isomorphism $\tilde{\omega} \colon \mathrm{Der}(A) \to \mathrm{DR}^1(A)$ defined by $\theta \mapsto i_\theta(\omega)$;
- symplectic derivations ($\mathcal{L}_\theta(\omega) = 0$) forming a Lie subalgebra $\mathrm{Der}_\omega(A) \subseteq \mathrm{Der}(A)$;
- map $\theta \colon \mathrm{DR}^0(A) \to \mathrm{Der}_\omega(A)$ given by $f \mapsto \tilde{\omega}^{-1}(\mathrm{d}f)$ ("Hamiltonian derivation" associated to f);
- θ is symplectic iff $i_\theta(\omega)$ is closed in $\mathrm{DR}^1(A)$, hamiltonian iff $i_\theta(\omega)$ is exact in $\mathrm{DR}^1(A)$;
- Poisson brackets $\{f, g\} := i_{\theta_f} i_{\theta_g}(\omega)$ that make $\mathrm{DR}^0(A)$ a Lie algebra;
- the map θ fits into an exact sequence of Lie algebras $0 \to H^0(\mathrm{DR}^\bullet(A)) \to \mathrm{DR}^0(A) \to \mathrm{Der}_\omega(A) \to H^1(\mathrm{DR}^\bullet(A)) \to 0$.

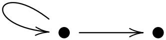
Quivers by example

A quiver is just a directed graph with no constraints on loops and multiple arcs.



Quivers by example

A quiver is just a directed graph with no constraints on loops and multiple arcs.

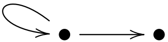


A representation of a quiver is given by the choice of a (f.d.) vector space for each vertex and a linear map for each arrow.

$$\mathrm{Rep}(Q,(d_1,d_2)) = \mathsf{Mat}_{d_1,d_1}(\mathbb{C}) \oplus \mathsf{Mat}_{d_1,d_2}(\mathbb{C})$$

Quivers by example

A quiver is just a directed graph with no constraints on loops and multiple arcs.



A representation of a quiver is given by the choice of a (f.d.) vector space for each vertex and a linear map for each arrow.

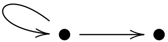
$$\mathrm{Rep}(Q, (d_1, d_2)) = \mathsf{Mat}_{d_1,d_1}(\mathbb{C}) \oplus \mathsf{Mat}_{d_1,d_2}(\mathbb{C})$$

On $\mathrm{Rep}(Q, \vec{d})$ there is the action of an algebraic group $G_{\vec{d}}$ by "change of basis".

$$G_{(d_1,d_2)} = (\mathsf{GL}_{d_1}(\mathbb{C}) \times \mathsf{GL}_{d_2}(\mathbb{C}))/\mathbb{C}^*$$

Quivers by example

A quiver is just a directed graph with no constraints on loops and multiple arcs.



A representation of a quiver is given by the choice of a (f.d.) vector space for each vertex and a linear map for each arrow.

$$\mathrm{Rep}(Q, (d_1, d_2)) = \mathsf{Mat}_{d_1,d_1}(\mathbb{C}) \oplus \mathsf{Mat}_{d_1,d_2}(\mathbb{C})$$

On $\mathrm{Rep}(Q, \vec{d})$ there is the action of an algebraic group $G_{\vec{d}}$ by "change of basis".

$$G_{(d_1,d_2)} = (\mathsf{GL}_{d_1}(\mathbb{C}) \times \mathsf{GL}_{d_2}(\mathbb{C}))/\mathbb{C}^*$$

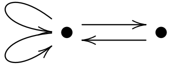
The corresponding quotient,

$$\mathcal{R}(Q, \vec{d}) = \mathrm{Rep}(Q, \vec{d})/G_{\vec{d}}$$

is the *moduli space of representations of* Q *with dim. vector* $\vec{d}$.

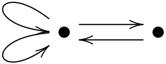
Where does symplectic geometry enter the picture?

To each quiver Q we can associate its *double* $\overline{Q}$, by adding to each arrow a in Q an arrow a^* that goes in the opposite direction.



Where does symplectic geometry enter the picture?

To each quiver Q we can associate its *double* $\overline{Q}$, by adding to each arrow a in Q an arrow a^* that goes in the opposite direction.



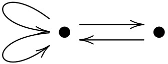
Its representation space can be seen as a cotangent bundle:

$$\mathrm{Rep}(\overline{Q}, \vec{d}) = T^*\mathrm{Rep}(Q, \vec{d})$$

thus it is equipped with a canonical symplectic form.

Where does symplectic geometry enter the picture?

To each quiver Q we can associate its *double* $\overline{Q}$, by adding to each arrow a in Q an arrow a^* that goes in the opposite direction.



Its representation space can be seen as a cotangent bundle:

$$\mathrm{Rep}(\overline{Q}, \vec{d}) = T^*\mathrm{Rep}(Q, \vec{d})$$

thus it is equipped with a canonical symplectic form. Moreover, the action of $G_{\vec{d}}$ on $\mathrm{Rep}(\overline{Q}, \vec{d})$ is just the lift of its action on the base, hence it is Hamiltonian and has a moment map

$$\mu\colon \mathrm{Rep}(\overline{Q}, \vec{d}) \to \mathfrak{g}_{\vec{d}}$$

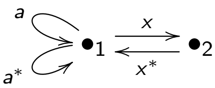
By judiciously chosing a point in $\mathfrak{g}_{\vec{d}}$, we can obtain some interesting (smooth, symplectic) varieties via Marsden-Weinstein reduction.

Where does n.c. symplectic geometry enter the picture?

On the other hand, there is a noncommutative algebra naturally associated with $\overline{Q}$, its path algebra $\mathbb{C}\overline{Q}$. It is defined as the $\mathbb{C}$-vector space having as basis the set of *paths* in $\overline{Q}$, with product given by composition of paths (or 0 if the paths do not compose).

Where does n.c. symplectic geometry enter the picture?

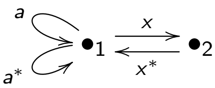
On the other hand, there is a noncommutative algebra naturally associated with $\overline{Q}$, its path algebra $\mathbb{C}\overline{Q}$. It is defined as the $\mathbb{C}$-vector space having as basis the set of *paths* in $\overline{Q}$, with product given by composition of paths (or 0 if the paths do not compose).



$$\mathbb{C}\overline{Q} = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$$
$$A_{11} \simeq \mathbb{C}\langle a, a^*, x^*x \rangle,$$
etc.

Where does n.c. symplectic geometry enter the picture?

On the other hand, there is a noncommutative algebra naturally associated with $\overline{Q}$, its path algebra $\mathbb{C}\overline{Q}$. It is defined as the $\mathbb{C}$-vector space having as basis the set of *paths* in $\overline{Q}$, with product given by composition of paths (or 0 if the paths do not compose).



$$\mathbb{C}\overline{Q} = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$$
$$A_{11} \simeq \mathbb{C}\langle a, a^*, x^* x\rangle,$$
etc.

The equivalence class in $\mathrm{DR}^2(\mathbb{C}\overline{Q})$ of the 2-form

$$\omega_{\mathrm{nc}} := \sum_{a \in Q} \mathrm{d}a\, \mathrm{d}a^*$$

endows $\mathbb{C}\overline{Q}$ with a non-commutative symplectic structure.

Gibbons-Hermsen varieties

Take $r, n \in \mathbb{N}$ and

$$\begin{aligned} V_{n,r} &= \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C}) \oplus \mathrm{Mat}_{r,n}(\mathbb{C}) \\ &= T^*(\mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C})) \end{aligned}$$

Gibbons-Hermsen varieties

Take $r, n \in \mathbb{N}$ and

$$\begin{aligned} V_{n,r} &= \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C}) \oplus \mathrm{Mat}_{r,n}(\mathbb{C}) \\ &= T^*(\mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C})) \end{aligned}$$

We consider the action of $\mathrm{GL}_n(\mathbb{C})$ on $V_{n,r}$ given by

$$G.(X, Y, v, w) = (GXG^{-1}, GYG^{-1}, Gv, wG^{-1})$$

Hamiltonian action with moment map $V_{n,r} \to \mathfrak{gl}_n(\mathbb{C})$

$$\mu(X, Y, v, w) = [X, Y] + vw$$

Gibbons-Hermsen varieties

Take $r, n \in \mathbb{N}$ and

$$\begin{aligned} V_{n,r} &= \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C}) \oplus \mathrm{Mat}_{r,n}(\mathbb{C}) \\ &= T^*(\mathrm{Mat}_{n,n}(\mathbb{C}) \oplus \mathrm{Mat}_{n,r}(\mathbb{C})) \end{aligned}$$

We consider the action of $\mathrm{GL}_n(\mathbb{C})$ on $V_{n,r}$ given by

$$G.(X, Y, v, w) = (GXG^{-1}, GYG^{-1}, Gv, wG^{-1})$$

Hamiltonian action with moment map $V_{n,r} \to \mathfrak{gl}_n(\mathbb{C})$

$$\mu(X, Y, v, w) = [X, Y] + vw$$

For every $\tau \in \mathbb{C}^*$ the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mu^{-1}(\tau I)$ is free, so by Marsden-Weinstein we have a smooth symplectic manifold of dimension $2nr$

$$\mathcal{C}_{n,r} = \mu^{-1}(\tau I)/\,\mathrm{GL}_n(\mathbb{C})$$

The $r = 1$ case

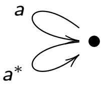
$$G_{(n,1)} = (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}))/\mathbb{C}^* \simeq \mathrm{GL}_n(\mathbb{C})$$

The $r = 1$ case

$$G_{(n,1)} = (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}))/\mathbb{C}^* \simeq \mathrm{GL}_n(\mathbb{C})$$

It turns out that in this case we can use a simpler quiver

$\overline{Q}_\circ =$



$$\mathbb{C}\overline{Q}_\circ = \mathbb{C}\langle a, a^* \rangle$$

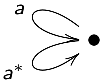
Then $\mathrm{DR}^0(\mathbb{C}\overline{Q}_\circ)$ is the vector space of necklace words in a and a^*, with bracket given by

$$\{f, g\} = \frac{\partial f}{\partial a}\frac{\partial g}{\partial a^*} - \frac{\partial f}{\partial a^*}\frac{\partial g}{\partial a} \mod [\mathbb{C}\overline{Q}_\circ, \mathbb{C}\overline{Q}_\circ]$$

The $r = 1$ case

$$G_{(n,1)} = (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}))/\mathbb{C}^* \simeq \mathrm{GL}_n(\mathbb{C})$$

It turns out that in this case we can use a simpler quiver

$\overline{Q}_\circ =$ 

$$\mathbb{C}\overline{Q}_\circ = \mathbb{C}\langle a, a^* \rangle$$

Then $\mathrm{DR}^0(\mathbb{C}\overline{Q}_\circ)$ is the vector space of necklace words in a and a^*, with bracket given by

$$\{f, g\} = \frac{\partial f}{\partial a}\frac{\partial g}{\partial a^*} - \frac{\partial f}{\partial a^*}\frac{\partial g}{\partial a} \mod [\mathbb{C}\overline{Q}_\circ, \mathbb{C}\overline{Q}_\circ]$$

Theorem (Ginzburg 2000)

Each symplectic variety $\mathcal{C}_{n,1}$ can be embedded as a coadjoint orbit in the Lie algebra $\mathrm{DR}^0(\mathbb{C}\overline{Q}_\circ)$.

Calogero-Moser correspondence

Theorem (Berest, Wilson 1999)

The space $\mathcal{C} := \bigsqcup_{n \in \mathbb{N}} \mathcal{C}_{n,1}$ *parametrizes isomorphism classes of right ideals in the first Weyl algebra over* $\mathbb{C}$

$$A_1 = \mathbb{C}\langle a, a^* \rangle / (aa^* - a^*a - 1)$$

Moreover, the natural action of the group $\operatorname{Aut} A_1$ *on each of the* $\mathcal{C}_{n,1}$ *is transitive.*

Calogero-Moser correspondence

Theorem (Berest, Wilson 1999)

The space $\mathcal{C} := \bigsqcup_{n\in\mathbb{N}} \mathcal{C}_{n,1}$ parametrizes isomorphism classes of right ideals in the first Weyl algebra over $\mathbb{C}$

$$A_1 = \mathbb{C}\langle a, a^*\rangle/(aa^* - a^*a - 1)$$

Moreover, the natural action of the group $\operatorname{Aut} A_1$ on each of the $\mathcal{C}_{n,1}$ is transitive.

This was proved by BW in a very roundabout way. But:

$$\operatorname{Aut} A_1 \simeq \operatorname{Aut}(\mathbb{C}\overline{Q}_\circ; [a, a^*])$$

the group of symplectic automorphisms of $\mathbb{C}\overline{Q}_\circ$ (i.e., algebra automorphisms of $\mathbb{C}\overline{Q}_\circ$ preserving $[a, a^*]$).

Calogero-Moser correspondence

Theorem (Berest, Wilson 1999)

The space $\mathcal{C} := \bigsqcup_{n\in\mathbb{N}} \mathcal{C}_{n,1}$ parametrizes isomorphism classes of right ideals in the first Weyl algebra over $\mathbb{C}$

$$A_1 = \mathbb{C}\langle a, a^* \rangle / (aa^* - a^*a - 1)$$

Moreover, the natural action of the group $\operatorname{Aut} A_1$ on each of the $\mathcal{C}_{n,1}$ is transitive.

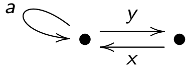
This was proved by BW in a very roundabout way. But:

$$\operatorname{Aut} A_1 \simeq \operatorname{Aut}(\mathbb{C}\overline{Q}_\circ; [a, a^*])$$

the group of symplectic automorphisms of $\mathbb{C}\overline{Q}_\circ$ (i.e., algebra automorphisms of $\mathbb{C}\overline{Q}_\circ$ preserving $[a, a^*]$).
So from this perspective it is a natural result after all.

A result of Bielawski and Pidstrygach

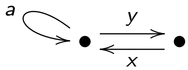
Next simplest case: $r = 2$. Taking

$Q_{BP} =$ 

we can embed $V_{n,2}$ into $\mathrm{Rep}(\overline{Q}_{BP}, (n, 1))$.

A result of Bielawski and Pidstrygach

Next simplest case: $r = 2$. Taking

$Q_{BP} =$ 

we can embed $V_{n,2}$ into $\mathrm{Rep}(\overline{Q}_{BP}, (n, 1))$.

Theorem (Bielawski, Pidstrygach 2008)

The group $\mathrm{TAut}(\mathbb{C}\overline{Q}_{BP}; c)$ of (tame) symplectic automorphisms of $\mathbb{C}\overline{Q}_{BP}$ acts transitively on $\mathcal{C}_{n,2}$.

Here $c = [a, a^*] + [x, x^*] + [y, y^*]$.

Some work in progress

It is known from work of Baranowski-Ginzburg-Kustnezov that $\bigsqcup_{n\in\mathbb{N}} \mathcal{C}_{n,r}$ parametrizes isomorphism classes of a certain class of right sub-A_1-modules in B_1^r.

Some work in progress

It is known from work of Baranowski-Ginzburg-Kustnezov that $\coprod_{n\in\mathbb{N}} \mathcal{C}_{n,r}$ parametrizes isomorphism classes of a certain class of right sub-A_1-modules in B_1^r.

Conjecture (G. Wilson): on $\mathcal{C}_{n,r}$ there is a transitive action of

$$\Gamma_{\mathsf{sc}}^{\mathrm{alg}} \times \mathrm{PGL}_r(\mathbb{C}[z])$$

(where $\Gamma_{\mathsf{sc}}^{\mathrm{alg}}$ is a group of maps of the form $e^p I_r$ for some polynomial $p \in z\mathbb{C}[z]$).

Some work in progress

It is known from work of Baranowski-Ginzburg-Kustnezov that $\bigsqcup_{n\in\mathbb{N}} \mathcal{C}_{n,r}$ parametrizes isomorphism classes of a certain class of right sub-A_1-modules in B_1^r.

Conjecture (G. Wilson): on $\mathcal{C}_{n,r}$ there is a transitive action of

$$\Gamma_{\mathsf{sc}}^{\mathrm{alg}} \times \mathsf{PGL}_r(\mathbb{C}[z])$$

(where $\Gamma_{\mathsf{sc}}^{\mathrm{alg}}$ is a group of maps of the form $e^p I_r$ for some polynomial $p \in z\mathbb{C}[z]$).

Theorem (A.T., I. Mencattini)

When $r = 2$ there is a morphism of groups

$$i\colon \Gamma^{\mathrm{alg}} \to \mathrm{PTAut}(\mathbb{C}\overline{Q}_{BP}; c)$$

and the induced action on $\mathcal{C}_{n,2}$ is transitive (at least) on a dense open subset.