The moduli space of linear control systems

Alberto Tacchella
altacch@gmail.com

Shapes of Thought
Geometry, mathematical physics, and philosophy
In honour of Claudio Bartocci's 60th birthday
October 27, 2022

Outline of the talk

1. Linear control systems
2. Interlude: Quivers and their representations
3. Moduli spaces for LTI systems

A clash of cultures

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It turns out that engineers usually think about dynamical systems in a different way:

- input signal $u\colon I \to \mathbb{R}^m$ $(I \subseteq \mathbb{R})$
- output signal $y\colon I \to \mathbb{R}^p$
- "device" mapping each input signal to an output signal.

Linear control systems

Under suitable hypotheses (including linearity and time invariance), we can convert such a description into a *state space realization*:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

where $x \in \mathbb{R}^n$ is the *state vector*, $u \in \mathbb{R}^m$ is the (instantaneous) *input*, and $y \in \mathbb{R}^p$ is the (instantaneous) *output* of the system. Here $A \in \mathrm{Mat}_{n,n}(\mathbb{R})$, $B \in \mathrm{Mat}_{n,m}(\mathbb{R})$, $C \in \mathrm{Mat}_{p,n}(\mathbb{R})$, $D \in \mathrm{Mat}_{p,m}(\mathbb{R})$.

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There is an underlying "change of paradigm" here: from *closed* to *open* systems.

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Open systems are characterized by the presence of *interfaces* that let you *compose* them (usually in multiple ways).
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The study of open systems using tools from category theory became a popular topic in the last few years, and can shed a new light even on old and venerable subjects like classical mechanics[1]...

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...but this is not the topic of this talk.

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The nature of state

Let us return to our state space realization:

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represents the same system as realization ($\dagger$) whenever

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Let us call an equivalence class of quadruples (A, B, C, D) under the above identification an *LTI system of signature* (m, n, p).

Controllability

Let us denote by $\Phi \colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the flow map of an LTI system. *Lagrange's formula* tells us that

$$\Phi_{t_0,x_0}(t) = \mathrm{e}^{A(t-t_0)}x_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

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An LTI system is *(completely) controllable* if for every $x_0 \in \mathbb{R}^n$ there exist $t > 0 \in \mathbb{R}$ and an input signal $u\colon [0, t] \to \mathbb{R}^m$ such that $\Phi_{0,x_0}(t) = 0$.

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Theorem

An LTI system $[A, B, C, D]$ is controllable iff the block matrix

$$\Gamma_{\mathrm{ctr}} := \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

has full rank ($= n$).

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An LTI system $[A, B, C, D]$ is observable iff the block matrix

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has full rank ($= n$)*.*

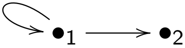
These results look very similar! Indeed there is a *duality map*

$$(A, B, C, D) \mapsto (A^\top, C^\top, B^\top, D^\top)$$

which exchanges controllability and observability.

Obligatory slides on quiver representations (1)

A *quiver* is a directed graph (possibly with loops or multiple arcs)



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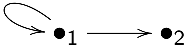

A *representation* of a quiver Q over the field $\mathbb{K}$ is given by the choice of a (finite dimensional) vector space for each vertex and a linear map for each arrow

$$\mathrm{Rep}(Q, \vec{d}) = \mathrm{Mat}_{d_1,d_1}(\mathbb{K}) \oplus \mathrm{Mat}_{d_1,d_2}(\mathbb{K})$$

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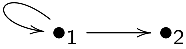

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Two representations of Q with dimension vector $\vec{d}$ are *isomorphic* if they are related by a family of linear isomorphisms (one for each vertex), or equivalently by the action of an algebraic group $G_{\vec{d}}$

$$G_{\vec{d}} = (\mathrm{GL}_{d_1}(\mathbb{K}) \times \mathrm{GL}_{d_2}(\mathbb{K}))/\mathbb{K}^*$$
$$(g_1, g_2).(A, B) = (g_1 A g_1^{-1}, g_1 B g_2^{-1})$$

Obligatory slides on quiver representations (2)

Goal: classify the isomorphism classes of representations of a quiver Q (for any given dimension vector). Some definitions:

1. Q is of *finite type* if it has only finitely many isomorphism classes of indecomposable representations.

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Trichotomy theorem: A connected quiver which is not of finite type is either tame or wild.

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Unfortunately, the quivers we shall be interested in are typically wild. The goal then becomes getting a handle on the quotient space

$$\mathrm{Rep}(Q, \vec{d})/G_{\vec{d}}$$

which is in general quite badly behaved (e.g. not Hausdorff).

Moduli spaces of quiver representations (1)

There are many ways to build "good" quotients. One option is simply to take the *categorical quotient*, i.e. the affine scheme corresponding to the ring of invariants:

$$\mathrm{Rep}(Q, \vec{d}) /\!\!/ G_{\vec{d}} := \operatorname{Spec} \mathbb{K}[\mathrm{Rep}(Q, \vec{d})]^{G_{\vec{d}}}$$

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The basic results in this case are:

- Each fiber of the projection map $\pi \colon \operatorname{Rep}(Q, \vec{d}) \to \operatorname{Rep}(Q, \vec{d}) /\!\!/ G_{\vec{d}}$ contains a unique *closed* orbit.
- (M. Artin) A point of $\operatorname{Rep}(Q, \vec{d})$ belongs to a closed orbit iff the corresponding representation is *semisimple*.

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Thus the affine variety $\mathrm{Rep}(Q,\vec{d})/\!\!/G_{\vec{d}}$ parametrizes isomorphism classes of semisimple representations of Q with dimension vector $\vec{d}$.

Moduli spaces of quiver representations (2)

Another, more "fine-grained" approach involves trading *invariants* for *semi-invariants*, that is functions f such that

$$f(g.v) = \chi(g) f(x)$$

for some (fixed) character $\chi \colon G_{\vec{d}} \to \mathbb{K}^*$.

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$$\chi_{\vec{\theta}}(g) = \prod_{i \in Q_0} (\det g_i)^{\theta_i}$$

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for some $\vec{\theta} \in \mathbb{Z}^{Q_0}$ such that $\vec{\theta} \cdot \vec{d} = 0$. One can define a notion of $\vec{\theta}$*-(semi)stability* for the points of $\mathrm{Rep}(Q, \vec{d})$, and using standard algebro-geometric constructions we get a quasi-projective variety $\mathcal{M}(Q, \vec{d}, \vec{\theta})$ and a quotient map

$$\pi \colon \mathrm{Rep}(Q, \vec{d})^{ss} \to \mathcal{M}(Q, \vec{d}, \vec{\theta})$$

with the stable points as an open subset on which $G_{\vec{d}}$ acts freely.

Moduli spaces for LTI systems

Consider again the data defining (a state space realization of) an LTI system of signature (m, n, p): a quadruple

$$(A, B, C, D) \in \mathrm{Mat}_{n,n}(\mathbb{R}) \times \mathrm{Mat}_{n,m}(\mathbb{R}) \times \mathrm{Mat}_{p,n}(\mathbb{R}) \times \mathrm{Mat}_{p,m}(\mathbb{R})$$

subject to the $\mathrm{GL}_n(\mathbb{R})$ action

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We start by getting rid of the D matrix, which is fixed by the GL_n action:

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So we can fix $D = 0$ ("strictly proper systems") and focus on the quotient

$$V_{m,n,p}/\,\mathrm{GL}_n(\mathbb{R})$$

Quiver interpretation of LTI systems

This can be reinterpreted as a moduli problem for (real) representations of a certain quiver Q... which however is not the "obvious" one.

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Problem: the group acting is "too large":

$$G_{(m,n,p)} = (\mathrm{GL}_m(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_p(\mathbb{R}))/\mathbb{R}^*$$
$$(g_1, g_2, g_3).(A, B, C) = (g_2 A g_2^{-1}, g_2 B g_1^{-1}, g_3 C g_2^{-1})$$

Change of basis in the input and output spaces *should not* be allowed!

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with dimension vector $\vec{d} = (1, n, 1)$.

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$$Q_{m,p} = \begin{tikzcd}\bullet_1 \arrow[r, Rightarrow, "m"] & \bullet_2 \arrow[loop above] \arrow[r, Rightarrow, "p"] & \bullet_3\end{tikzcd}$$

with dimension vector $\vec{d} = (1, n, 1)$.
Problem: the group naturally acting on $\mathrm{Rep}(Q, \vec{d})$ is still not quite right:

$$G_{(1,n,1)} = (\mathbb{R}^* \times \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^*)/\mathbb{R}^*$$
$$(\alpha, g, \beta).(A, B_{*j}, C_{i*}) = (gAg^{-1}, \alpha^{-1}gB_{*j}, \beta C_{i*}g^{-1})$$

The quotient eats up only one of the two unwanted parameters $\alpha, \beta \in \mathbb{R}^*$.

Quiver interpretation of LTI systems

This can be reinterpreted as a moduli problem for (real) representations of a certain quiver Q... which however is not the "obvious" one.

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "B"] & \mathbb{R}^n \arrow[loop above, "A"] \arrow[r, "C"] & \mathbb{R}^p
\end{tikzcd}$$

Third try: take

$$\begin{tikzcd}
Q_{m,p} = \bullet_1 \arrow[r, Rightarrow, bend left, "m"] & \bullet_2 \arrow[l, Rightarrow, bend left, "p"] \arrow[loop right]
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~~Problem~~: This time everything is right:

$$G_{(1,n)} = (\mathbb{R}^* \times \mathsf{GL}_n(\mathbb{R}))/\mathbb{R}^* \simeq \mathsf{GL}_n(\mathbb{R})$$
$$(\alpha, g).(A, B_{*j}, C_{i*}) = (gAg^{-1}, \alpha^{-1}gB_{*j}, \alpha C_{i*}g^{-1})$$

Note: $Q_{m,p}$ is an example of a "generalized framed quiver", as introduced in Bartocci-Lanza-Rava, J. Geom. Phys. 118 (2017) 20.

The moduli space of minimal LTI systems

Let us call an LTI system *minimal* if it is both controllable and observable.

Theorem

The LTI system realized by a point $(A, B, C) \in V_{m,n,p}$ *is minimal iff* $(A, B_{*j}, C_{i*}) \in \mathrm{Rep}(Q_{m,p}, \vec{d})$ *is a simple representation.*

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Proof sketch: a proper subrepresentation of dimension vector $(1, k)$ with $k < n$ would imply $\operatorname{rank} \Gamma_{\mathrm{ctr}} \leq k$, contradicting controllability; whereas a proper subrepresentation of dimension vector $(0, k)$ with $0 < k < n$ would imply $\operatorname{rank} \Gamma_{\mathrm{obs}} \leq n - k$, contradicting observability.

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It is a general fact that the isomorphism classes of simple representations form a Zariski open smooth subvariety of $\mathrm{Rep}(Q_{m,p}, \vec{d}) /\!\!/ G_{\vec{d}}$. Moreover,

$$\dim(\mathrm{Rep}(Q_{m,p}, \vec{d}) /\!\!/ G_{\vec{d}}) = 1 - R_{Q_{m,p}}(\vec{d}, \vec{d})$$

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where $R_{Q_{m,p}}$ is the *(Euler-)Ringel form* of the quiver $Q_{m,p}$. In our case $R_{Q_{m,p}} = \left(\begin{smallmatrix} 1 & -m \\ -p & 0 \end{smallmatrix}\right)$, we thus recover a theorem of Hazewinkel:

Theorem

The moduli space $\mathcal{M}^{\min}_{m,n,p}$ *of minimal LTI systems is a smooth quasi-affine variety of dimension* $(m + p)n$*.*

The moduli space of controllable/observable systems

Consider now the possible stability parameters $\vec{\theta}$ for $Q_{m,p}$. Since $\vec{d} = (1, n)$ and we must have $\vec{\theta} \cdot \vec{d} = 0$, there are only two possible choices:

$$\vec{\theta}_+ := (-n, 1) \qquad \vec{\theta}_- := (n, -1)$$

up to a common (positive) factor.

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1. *controllable iff $(A, B_{*j}, C_{i*}) \in \mathrm{Rep}(Q_{m,p}, \vec{d})$ is $\vec{\theta}_+$-stable;*
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We thus have isomorphisms

$$\mathcal{M}^{\mathrm{ctr}}_{m,n,p} \simeq \mathcal{M}(Q_{m,p}, \vec{d}, \vec{\theta}_+) \qquad \mathcal{M}^{\mathrm{obs}}_{m,n,p} \simeq \mathcal{M}(Q_{m,p}, \vec{d}, \vec{\theta}_-)$$

as smooth quasi-projective varieties of dimension $(m+p)n$.

Grassmannian embedding

Let us denote by $\mathsf{Gr}_k(\infty)$ the Grassmannian of k-dimensional subspaces in a linear space of countable dimension.

Theorem (Le Bruyn, Reineke)

There is an embedding $\bigsqcup_n \mathcal{M}^{\mathrm{ctr}}_{m,n,p} \to \mathsf{Gr}_{m+p}(\infty)$ *whose image is the union of all the affine cells* X_I *over the multi-index sets* $I = (i_1, \ldots, i_{m+p})$ *such that* $\{m+1, \ldots, m+p, m+p+n\} \subseteq I$*.*

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By duality we have an analogous embedding $\bigsqcup_n \mathcal{M}^{\mathrm{obs}}_{m,n,p} \to \mathsf{Gr}_{m+p}(\infty)$; its image is again an union of affine cells X_I, this time characterized by the condition $\{1, \ldots, m, m+p+n\} \subseteq I$.

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Note: in arXiv:1509.00749 Le Bruyn has used these results to argue that these moduli spaces are defined over $\mathbb{F}_1$, and are related to $\operatorname{Spec} \mathbb{Z}$ over that base (at least using one of the many competing approaches to $\mathbb{F}_1$).

A special case: SISO systems

The case $m = p = 1$ is exceptional: indeed in that case one has

$$\mathcal{M}^{\mathrm{ctr}}_{1,n,1} \simeq \mathbb{A}^{2n}$$

and the quotient map $V_{1,n,1} \to \mathcal{M}^{\mathrm{ctr}}_{1,n,1}$ has a continuous (even algebraic) section, given by the well known *canonical control form*

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

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Hazewinkel: $\mathcal{M}^{\mathrm{ctr}}_{m,n,p}$ is *never* projective when $m > 1$. It follows that there are no *global algebraic* canonical forms for LTI systems with more than one input.

A final message

Dear Claudio,
thank you very much for all the stuff I learned from you in the past 16 years of (more or less frequent) interactions, and...

Happy birthday!