Università degli studi di Genova

Facoltà di Scienze Matematiche, Fisiche e Naturali
Dipartimento di Matematica

Tesi di Dottorato di Ricerca in Matematica
XXI ciclo

A multicomponent generalization of the KP/CM correspondence

Candidato:
Alberto Tacchella

Relatore:
Prof. **Claudio Bartocci**
Dipartimento di Matematica
Università degli Studi di Genova

Settore scientifico-disciplinare: MAT/07

Contents

Introduction

It seems appropriate to begin a Thesis devoted to a generalization of the KP/CM correspondence by explaining what the KP/CM correspondence is. First of all, let's expand the two acronyms:

- KP stands for "Kadomtsev-Petviashvili" and refers to the following nonlinear partial differential equation[1]:

$$\frac{3}{4}u_{yy} = (u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u_x u)_x \tag{1}$$

  This equation was first written down in 1970 by Kadomtsev and Petviashvili [19] as a two-dimensional version of the popular KdV (Korteweg-de Vries) equation; the idea was to study the stability properties of one-dimensional solitons against transverse perturbations. It turns out that equation (1), just like her little sister KdV, is completely integrable and in fact is today recognized as one of the most important examples of an integrable PDE in $2+1$ dimensions.

- CM stands for "Calogero-Moser" and refers to a wide class of (finite-dimensional) completely integrable dynamical systems. Here we consider only the very first of these models, nowadays called the *rational Calogero-Moser system*, which is defined by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{g^2}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{1}{(q_i - q_j)^2} \tag{2}$$

  where q_i, p_i are the canonical coordinates associated to a set of n point particles of unit mass moving on a line and g is a coupling constant. This Hamiltonian system surfaced in 1969 [8, 9] as a *quantum-mechanical* model whose main interest was its explicit solvability; this was achieved by Calogero in [10]. Noting that the model is "integrable" in the quantum sense (i.e., one can write down a maximal set of mutually commuting observables), he conjectured that the same result should hold for the classical version as well. This suggestion was taken up by Moser, who in [29] managed to put the equations of motion of the system in Lax form and prove their integrability. From that point on, this system has been extensively studied and

[1] Here and elsewhere, as usual in the integrable system literature, a notation such as u_x stands for the partial derivative $\frac{\partial u}{\partial x}$.

generalized in every conceivable direction: see e.g. the celebrated review papers by Olshanetsky and Perelomov [31, 32] dealing with the classical and the quantum models, respectively.

By the phrase "KP/CM correspondence" we refer to the following fact, discovered by Krichever in [22]. Let $u = u(x, y, t)$ be a solution of the KP equation (1) that is a *rational* function of x vanishing for $x \to \infty$. One can readily verify that such a solution must be of the form

$$u(x, y, t) = -2 \sum_{j=1}^{n} (x - x_j(y, t))^{-2} \tag{3}$$

for some $n \in \mathbb{N}$ and some functions $x_1, \ldots, x_n$ of y and t. Now suppose the x_j's describe the position of n particles on a (complex) line, with the y variable serving as evolution parameter (i.e., "time") and the t variable frozen. Define the corresponding canonical momenta as $p_j := \frac{1}{2}\partial_y x_j$. Then we claim that those "particles" evolve exactly as the particles of a rational CM system, i.e. according to the Hamiltonian (2). Vice versa, if $\{x_i(t), p_i(t)\}_{i=1\ldots n}$ is any solution of a n-particle CM system then there exist a rational solution u of the KP equation whose poles in x evolve in the same manner.

Although already striking, this phenomenon is even more general. Both the KP equation and the CM system are the first non-trivial members of two corresponding *hierarchies*, respectively of (integrable) partial differential equations and of (integrable) finite-dimensional Hamiltonian systems. The KP/CM correspondence extends to the whole two hierarchies, as shown by Shiota [40]: indeed for every $n \geq 1$ we have the equalities

$$\frac{\partial}{\partial t_n} x_i = \frac{\partial H_n}{\partial p_i} \qquad \frac{\partial}{\partial t_n} p_i = -\frac{\partial H_n}{\partial x_i} \tag{4}$$

where t_n is the n-th time of the KP hierarchy and H_n is (up to scalar factors) the n-th Hamiltonian of the CM hierarchy.

A further step forward was taken in the paper [43] by George Wilson. Here the correspondence is motivated on geometrical grounds, in the following manner: the space of rational solutions of the KP hierarchy is a certain (infinite-dimensional) Grassmannian, called the *adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}$. On the other hand, the solutions of the CM hierarchy live as integral curves in a "completed" version of the Calogero-Moser phase space $\mathcal{C}$. Wilson defined a bijection $\beta \colon \mathcal{C} \to \mathrm{Gr}^{\mathrm{ad}}$ that intertwines the CM flows on $\mathcal{C}$ and the KP flows on $\mathrm{Gr}^{\mathrm{ad}}$, thereby giving an explicit identification between the geometry of the two systems (this construction will be explained at length in Chapter 1).

Before moving on, we would like to make the point that the history of the KP/CM correspondence is much more complicated than the short sketch we draw above. For example the first evidence of a connection between integrable systems of PDEs and finite-dimensional Hamiltonian systems was seen in 1977 by Airault, McKean and Moser [1]: motivated by the preliminary works [25, 41] they studied the rational, trigonometric and elliptic solutions of the KdV equation (notice that KdV is a reduction of KP) and found out that the motion of their poles is governed by the CM system of the corresponding flavour (with certain constraints). These early results were confirmed and deepened by

Krichever [22, 23], who established the relationship between rational (elliptic) solutions of the KP equation and the rational (elliptic) CM system. From that point on, the search for other correspondences of this kind has become one of the most active field in the theory of integrable systems.

We now return for good to the rational case. At the end of [43], Wilson suggests that a generalization of his approach to the multicomponent versions of the CM and KP systems should yield similar results, as indicated by some well-known calculations [13,24]; this Thesis aspires to be a first step in this direction.

The idea of generalizing the KP/CM correspondence to a multicomponent setting is not at all unprecedented; indeed we know of two different works related to this. First there are the papers [3, 4] by Baranovsky, Ginzburg and Kuznetsov; there they obtain (among other things) a bijection between a generalization of the adelic Grassmannian and the disjoint union of some quiver varieties. By taking a particular case, this construction gives a bijection between a multicomponent version of the adelic Grassmannian and the (completed) phase space of the multicomponent CM system. However, the connection with the corresponding dynamical systems is left completely unexplored (indeed, one of their goals seems to be *not* to use any result coming from integrable systems theory).

Then there are the papers [5, 6] by Ben-Zvi and Nevins (it is from here that we borrowed the term “KP/CM correspondence”). Their purpose is even more ambitious, since they treat simultaneously all the possible kinds of potentials (rational, trigonometric and elliptic); however their work is technically very demanding, hovering at a very abstract level, and it is difficult to extract concrete formulas from it.

Compared with these two approaches, our aim is much more modest: we simply introduce some plausible definitions for the multicomponent versions of the various Grassmannians parametrizing solutions of the KP hierarchy and study the rationality of the solutions coming from such spaces. Our main results are theorem 2.77, in which we prove such rationality properties for the solutions of the multicomponent KP hierarchy associated to what we call “1-point” multicomponent Grassmannians, and theorem 2.103, which shows that if we restrict ourselves to the *matrix KP hierarchy* (a sub-hierarchy of the full multicomponent KP) then *every* subspace in the multicomponent adelic Grassmannian (not only the ones whose support consists of a single point) gives a rational solution.

The geometric interpretation of the multicomponent KP/CM correspondence in subsection 2.4.4 is still at a very embryonal stage, mainly because of the lack of an explicit formula fulfilling the rôle that equations (1.94–1.95) play in the scalar setting (i.e., relating the Baker function and tau function of a point in the Grassmannian to the matrices that determine a point in the CM phase space).

The Thesis is structured as follows. In the first Chapter we review the rational KP/CM correspondence as developed in [43]; in the first three Sections we introduce the various objects involved (CM phase spaces, Grassmannians, KP hierarchy) and in Section 1.4 we quickly go through the main steps needed to define the map β described above and establish its properties. In this Chapter we took special care in making homogeneous and unambiguous the notation used for the various objects (often coming from different papers) and in clarifying some points usually left implicit in the existing literature.

In the second Chapter we confront the problem of generalizing the previous constructions to the multicomponent case. Also here a lot of material is well-known: among it the description of the multicomponent CM system in Section 2.1 and the definition of the multicomponent Segal-Wilson Grassmannian in Section 2.2. The new definitions are mainly contained in subsections 2.2.1 and 2.2.2. The heart of the Thesis is subsection 2.3.4, where we study the rationality of the solutions to the multicomponent KP hierarchy and prove the afore-mentioned theorem 2.77. Finally in Section 2.4 we show how to recover the matrix KP hierarchy from the general framework, prove theorem 2.103 and relate these results to a known calculation that links the motion of the poles of rational solutions to the matrix KP equation to the multicomponent CM system.

We would like to emphasize again that the results obtained are of a very preliminary nature, and that to really clarify the geometric nature of the multicomponent KP/CM correspondence much is left to do. For example we completely ignore the questions related to collisions of multicomponent CM particles that feature prominently in [43], neither we attack the problem of a complete characterization of rational solutions to the multicomponent KP hierarchy (they are surely not exhausted by the rather limited class described by theorem 2.77). Other hints for future developments may also come from a comparison of our approach with the algebraic theory of the multivariable KP hierarchy developed by the Russian school [33–35]; all these directions are promising sources for further work.

Chapter 1

The scalar KP/CM correspondence

In this Chapter we illustrate the scalar KP/CM correspondence. In Section 1.1 we describe the phase space of the complexified rational Calogero-Moser system by a process of symplectic reduction. In Section 1.2 we define various infinite-dimensional Grassmannians that play a rôle in parametrizing the solutions of the KP hierarchy. In Section 1.3 we recall the standard definition of the KP hierarchy in terms of pseudo-differential operators and the relationship between the points of the Grassmannians previously introduced and the corresponding solutions. Finally in Section 1.4 we define the map implementing the correspondence.

1.1 Calogero-Moser spaces

We define the (complexified) **rational Calogero-Moser system** as the Hamiltonian system consisting of n point particles of unit mass on the complex plane whose evolution is determined by the following Hamiltonian[1]:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{(q_i - q_j)^2} \tag{1.1}$$

Some care is needed in defining the configuration space for this system, since H is clearly singular when $q_i = q_j$ for some pair of distinct indices i and j. So let $\Delta \subseteq \mathbb{C}^n$ be the union of the $\binom{n}{2} = \frac{1}{2}n(n-1)$ hyperplanes defined by the equations $x_i = x_j$ for $i, j = 1 \dots n$, $i < j$ and put

$$\mathbb{C}^n_{\text{reg}} := \mathbb{C}^n \setminus \Delta \tag{1.2}$$

This is a (disconnected) open submanifold of $\mathbb{C}^n$ parametrizing n-tuples of distinct complex numbers, so the Hamiltonian (1.1) is everywhere defined on it. Furthermore, we consider the n particles as *indistinguishable*, i.e. we regard two configurations that differ only for the ordering of the particles as the same configuration. This corresponds to

[1]Notice that comparing with equation (2) in the Introduction we have fixed $g = -\mathrm{i}$ to get an *attractive* interaction between the particles; more about this choice in the following.

taking the quotient of $\mathbb{C}^n$ by the natural action of the symmetric group S_n, so we define

$$\mathbb{C}^{(n)}_{\mathrm{reg}} := \mathbb{C}^n_{\mathrm{reg}}/S_n \tag{1.3}$$

We thus get a connected smooth algebraic variety whose points parametrize the unordered n-tuples of distinct complex numbers (or, if you prefer, the subsets of $\mathbb{C}$ with cardinality n); this will be the configuration space for the system with Hamiltonian (1.1).

We will now see, following [43], how the corresponding phase space $T^*(\mathbb{C}^{(n)}_{\mathrm{reg}})$ may arise from a symplectic reduction process. This construction is the complex version of a well-known procedure to solve the (real) CM system that goes back in essence to Olshanetsky and Perelomov's "projection method" [30] (see also [36]), with later elaborations by Moser [29] and by Kazhdan, Kostant and Sternberg [21]. An important bonus of the complex case is that the construction itself suggests a candidate for a "completion" of the phase space, allowing collisions between particles to happen (and to be meaningfully interpreted).

For any natural number $n \in \mathbb{N}$ consider the complex vector space

$$U_n := \mathrm{End}(\mathbb{C}^n) \oplus \mathbb{C}^n$$

A point in U_n is specified by a pair (X, w) where X is a $n \times n$ matrix and w is a row vector of length n. By the usual canonical isomorphisms of $\mathrm{End}(\mathbb{C}^n)$ and $\mathbb{C}^n$ with their duals (using respectively the trace form and the standard bilinear form on $\mathbb{C}^n$), the cotangent bundle $V_n := T^*U_n$ can be identified with

$$V_n \cong \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{End}(\mathbb{C}^n) \oplus \mathbb{C}^n \oplus \mathbb{C}^n \tag{1.4}$$

so that a point in V_n may be taken to be a quadruple (X, Y, v, w) comprising two $n \times n$ matrices X and Y, a column vector v and a row vector w; the pair (Y, v) is to be thought of as a cotangent vector applied on the point $(X, w) \in U_n$. Evidently V_n is a complex vector space of dimension $2n^2 + 2n$.

The canonical symplectic form on V_n reads

$$\omega(X, Y, v, w) = \mathrm{tr}(\mathrm{d}Y \wedge \mathrm{d}X + \mathrm{d}v \wedge \mathrm{d}w) \tag{1.5}$$

We now consider the action of the group $\mathrm{GL}(n, \mathbb{C})$ on U_n given by

$$G.(X, w) := (GXG^{-1}, wG^{-1})$$

and lift it to the cotangent bundle V_n, obtaining

$$G.(X, Y, v, w) := (GXG^{-1}, GYG^{-1}, Gv, wG^{-1}) \tag{1.6}$$

As is well known (see e.g. [27]), such a lifted action is automatically Hamiltonian and the corresponding momentum map $J\colon V_n \to \mathfrak{gl}(n)$ is given by

$$J(X, Y, v, w) = [X, Y] - vw \tag{1.7}$$

We fix the point $-I \in \mathfrak{gl}(n)$ and take its inverse image[2]

$$\tilde{\mathcal{C}}_n := \{ (X, Y, v, w) \in V_n \mid [X, Y] - vw = -I \} \tag{1.8}$$

The chosen point $-I \in \mathfrak{gl}(n)$ is stabilized by the whole of $\mathrm{GL}(n, \mathbb{C})$, so that the usual Marsden-Weinstein procedure yields the reduced space

$$\mathcal{C}_n := \tilde{\mathcal{C}}_n / \mathrm{GL}(n, \mathbb{C}) \tag{1.9}$$

that we will call the **n-particle Calogero-Moser space**. The (disjoint) union of all these spaces,

$$\mathcal{C} := \coprod_{n \in \mathbb{N}} \mathcal{C}_n \tag{1.10}$$

will be called simply the **Calogero-Moser space**.

The following two results are proven in [43].

Theorem 1.11. *$\mathcal{C}_n$ is a smooth affine algebraic variety of dimension $2n$.*

Very briefly, the proof goes as follows. It can be shown that the momentum map (1.7) is a submersion on $\tilde{\mathcal{C}}_n$, hence by the implicit function theorem $\tilde{\mathcal{C}}_n$ is a smooth subvariety of V_n of dimension $(2n^2 + 2n) - n^2 = n^2 + 2n$; but the action of $\mathrm{GL}(n)$ on $\tilde{\mathcal{C}}_n$ is free, so that the quotient space $\mathcal{C}_n$ is a smooth affine variety of dimension $2n$.

Theorem 1.12. *$\mathcal{C}_n$ is irreducible.*

We don't enter the proof of this fact, but we extract from it a lemma that will be useful later. First, a definition:

Definition 1.13. Let $q = (q_1, \dots, q_n)$ be a n-tuple of distinct complex numbers. A $n \times n$ matrix Y will be called a **Moser matrix associated to** q if, for every pair of indices $i, j = 1 \dots n$ with $i \neq j$, the off-diagonal entry Y_{ij} is equal to $(q_i - q_j)^{-1}$.

The diagonal entries of a Moser matrix are left unconstrained. Matrices of this form first appeared in [29], whence the name.

Theorem 1.14. *Let $(X, Y, v, w) \in \tilde{\mathcal{C}}_n$ and suppose Y is diagonalizable. Then the eigenvalues of Y are distinct, and the $\mathrm{GL}(n)$-orbit of (X, Y, v, w) contains a representative such that:*
1) *Y is diagonal, say $Y = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$;*
2) *$w = v^\top = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}$;*
3) *X is a Moser matrix associated to $(\lambda_1, \dots, \lambda_n)$.*
Moreover such a representative is unique up to the action (1.6) of a permutation matrix.

[2] It is exactly at this point that we choose to deal with the *attractive* CM system; to get back the Hamiltonian (2) in the Introduction we should take the inverse image of the point $-\mathrm{i}gI \in \mathfrak{gl}(n)$.

This fact has a number of interesting consequences: for example it shows that not every $n \times n$ matrix can occur as the Y in a point of $\tilde{\mathcal{C}}_n$ (if it is diagonalizable, then it has to have distinct eigenvalues). Another important corollary is the following: denote by $\mathcal{C}''_n$ the subspace of $\mathcal{C}_n$ consisting of the points for which the matrix Y is diagonalizable. Then there exists a bijection

$$\mathcal{C}''_n \to \mathbb{C}^{(n)}_{\text{reg}} \times \mathbb{C}^n \tag{1.15}$$

defined by mapping (X, Y, v, w) to $\{\lambda_1, \dots, \lambda_n\}, (\alpha_1, \dots, \alpha_n)$, where $\alpha_i := X_{ii}$. Indeed acting with a permutation matrix on a quadruple (X, Y, v, w) of the form described by theorem 1.14 has the only effect of simultaneously shuffling the λ_i and the α_i, as it is immediate to check. So we conclude that a point of $\mathcal{C}''_n$ is uniquely determined by the n (unordered, distinct) eigenvalues of Y and the n (ordered and not necessarily distinct) complex numbers α_i.

Symmetrically we can also introduce the space $\mathcal{C}'_n$ as the subspace of $\mathcal{C}_n$ consisting of the points for which the matrix X is diagonalizable. Then we have the analogous result:

Theorem 1.16. *Let $(X, Y, v, w) \in \tilde{\mathcal{C}}_n$ and suppose X is diagonalizable. Then the eigenvalues of X are distinct, and the $\mathrm{GL}(n)$-orbit of (X, Y, v, w) contains a representative such that:*
1) *X is diagonal, say $X = \mathrm{diag}(x_1, \dots, x_n)$;*
2) *$v = w^\top = \begin{pmatrix}1 & \dots & 1\end{pmatrix}$;*
3) *Y is a Moser matrix associated to $(x_1, \dots, x_n)$.*
Moreover such a representative is unique up to the action (1.6) of a permutation matrix.

Exactly as above, we can define a bijection

$$\mathcal{C}'_n \to \mathbb{C}^{(n)}_{\text{reg}} \times \mathbb{C}^n \tag{1.17}$$

by $(X, Y, v, w) \mapsto \{x_1, \dots, x_n\}, (p_1, \dots, p_n)$, where $p_i := Y_{ii}$. We can use the coordinate system $\{x_i, p_i\}$ so defined to express the restriction to $\mathcal{C}'_n$ of the reduced symplectic form on $\mathcal{C}_n$; a simple calculation then yields

$$\omega(x, p) = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}x_i \tag{1.18}$$

so that $\mathcal{C}'_n$ is isomorphic, as a symplectic manifold, to the phase space $T^*\mathbb{C}^{(n)}_{\text{reg}}$ of the complexified rational CM system.

Now for each $k \geq 1$ consider the flows on V_n generated by the Hamiltonians $H_k := \mathrm{tr}(-Y)^k$; the corresponding equations of motion are trivially integrated to give

$$t_k.(X, Y, v, w) = (X + k t_k (-Y)^{k-1}, Y, v, w) \tag{1.19}$$

where t_k is the time variable corresponding to H_k. Clearly these flows preserve $\tilde{\mathcal{C}}_n$ and are $\mathrm{GL}(n)$-invariant, so we can project them down to $\mathcal{C}_n$ and, by restriction, on $\mathcal{C}'_n$. In particular for $k = 2$ we have

$$H_2 = \mathrm{tr}\, Y^2 = \sum_{i=1}^{n} p_i^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{1}{(q_i - q_j)^2}$$

which is, up to a factor of 2, the Hamiltonian (1.1). Generally, the H_k coincide (again up to constant factors) with the Hamiltonians of the CM hierarchy[3], so we conclude that the time evolution of the n-particle CM hierarchy is embedded in the symplectic manifold $\mathcal{C}_n$ by the flows (1.19) on its open dense subset $\mathcal{C}'_n$. Then it is natural to identify the bigger manifold $\mathcal{C}_n$ with a "partial compactification" of the CM phase space[4]; the CM flows smoothly extends to this completion, and this enables us to continue analytically the motion through possible collisions between the particles.

In the sequel we will denote by Γ the commutative group of all *finite* combinations of CM flows acting on $\mathcal{C}_n$; then Γ consists of all sequences $\boldsymbol{t} = \{t_k\}_{k\geq 1}$ of complex numbers almost all equal to zero, and its action on $\mathcal{C}_n$ is

$$\boldsymbol{t}.(X, Y, v, w) = (X + \sum_{k\geq 1} k t_k (-Y)^{k-1}, Y, v, w) \tag{1.20}$$

We remark that Γ may be seen as the (additive) group of polynomials with complex coefficients and zero constant term, the correspondence being given by $\boldsymbol{t} \mapsto p = \sum_{k\geq 1} t_k z^k$; with this identification in mind, the action (1.20) reads

$$p.(X, Y, v, w) = (X + p'(-Y), Y, v, w) \tag{1.21}$$

where p' denotes, as usual, the derivative dp/dz.

1.2 Some infinite-dimensional Grassmannians

In this Section we introduce various infinite-dimensional Grassmannians related to the solutions of the KP hierarchy. Our main references here are [39, 42, 43], although we will sometimes change the terminology or the notation employed for clarity's sake.

1.2.1 The rational Grassmannian

Let $\mathcal{R}$ be the space of rational functions on the complex projective line $\mathbb{C}P^1$. We denote by $\mathcal{P}$ the subspace of polynomials and by $\mathcal{R}_-$ the subspace of rational functions that vanish at infinity; then the polynomial division algorithm gives us the direct sum decomposition

$$\mathcal{R} = \mathcal{P} \oplus \mathcal{R}_- \tag{1.22}$$

with associated canonical projection maps $\pi_+ : \mathcal{R} \to \mathcal{P}$ and $\pi_- : \mathcal{R} \to \mathcal{R}_-$ (cfr. Appendix B). The **full rational Grassmannian** $\overline{\mathrm{Gr}}^{\mathrm{rat}}$ is the set of closed linear subspaces $W \subseteq \mathcal{R}$ for which there exist polynomials $p, q \in \mathcal{P}$ such that

$$p\mathcal{P} \subseteq W \subseteq q^{-1}\mathcal{P} \tag{1.23}$$

[3]These are classically defined as $H_k = \frac{1}{k} \operatorname{tr} Y^k$; of course the additional factor $(-1)^k k$ does not alter the integral curves of the system.

[4]The compactification is only a partial one because particles can still escape to infinity.

Equivalently we can require the existence of a single polynomial $p \in \mathcal{P}$ such that $p\mathcal{P} \subseteq W \subseteq p^{-1}\mathcal{P}$: indeed, if W satisfies (1.23) and m is the least common multiple of p and q, we have

$$m\mathcal{P} \subseteq p\mathcal{P} \subseteq W \subseteq q^{-1}\mathcal{P} \subseteq m^{-1}\mathcal{P}$$

Given $W \in \overline{\mathrm{Gr}}^{\mathrm{rat}}$, let us define a linear operator $p_+ \colon W \to \mathcal{P}$ by $p_+ := \pi_+|_W$; we claim that this is a Fredholm operator, i.e. it has finite-dimensional kernel and cokernel. To see this notice that by (1.23) we can write

$$W = W' \oplus p\mathcal{P} \tag{1.24}$$

where W' is a linear subspace of finite dimension in $\mathcal{R}$. Clearly p_+ acts as the identity on $p\mathcal{P}$, so

$$\ker p_+ = \ker p'_+ \tag{1.25}$$

(where we have defined $p'_+ := p_+|_{W'}$) and the latter kernel is clearly finite-dimensional. Writing similarly $\mathcal{P} = U \oplus p\mathcal{P}$ (with $\dim U = \deg p$) and noting that $\operatorname{im} p_+$ definitely contains the second direct summand of (1.24), we have

$$\operatorname{coker} p_+ = \frac{U \oplus p\mathcal{P}}{\operatorname{im} p_+} \cong \frac{U}{\operatorname{im} p'_+} \tag{1.26}$$

This is a quotient between finite-dimensional subspaces, so

$$\dim \operatorname{coker} p_+ = \deg p - \dim \operatorname{im} p'_+ \tag{1.27}$$

is again finite.

Let's define the **virtual dimension** of W, denoted $\operatorname{vdim} W$, as the index of the Fredholm operator p_+; then the equalities (1.25-1.27) and the rank-nullity theorem for p'_+ tell us that

$$\operatorname{vdim} W = \dim \ker p'_+ - (\deg p - \dim \operatorname{im} p'_+) = \dim W' - \deg p \tag{1.28}$$

We denote by $\mathrm{Gr}^{\mathrm{rat}}$ the part of $\overline{\mathrm{Gr}}^{\mathrm{rat}}$ made by subspaces of virtual dimension zero, and call it the **(index zero) rational Grassmannian**. Notice that for the time being we regard these Grassmannians simply as sets; a topology on them will be introduced in subsection 1.2.4.

Theorem 1.29. *A subspace $W \in \overline{\mathrm{Gr}}^{\mathrm{rat}}$ has virtual dimension zero if and only if the codimension of the inclusion $W \subseteq q^{-1}\mathcal{P}$ coincides with the degree of q.*

Proof. Conditions (1.23) may be rewritten as $qp\mathcal{P} \subseteq qW \subseteq \mathcal{P}$, so

$$\operatorname{codim}_{q^{-1}\mathcal{P}} W = \operatorname{codim}_{\mathcal{P}} qW = \dim \frac{\mathcal{P}}{qW}$$

Now if we quotient out from both spaces in $\mathcal{P}/qW$ the common subspace $qp\mathcal{P}$ we get an isomorphic linear space which is the quotient of two manifestly finite-dimensional subspaces:

$$\frac{\mathcal{P}}{qW} \cong \frac{\mathcal{P}/qp\mathcal{P}}{qW/qp\mathcal{P}}$$

Moreover $qW/qp\mathcal{P} \cong W/p\mathcal{P} = W'$, so the desired codimension is (using (1.28))

$$\deg q + \deg p - \dim W' = \deg q - \operatorname{vdim} W$$

i.e. $\operatorname{vdim} W = \deg q - \operatorname{codim}_{q^{-1}\mathcal{P}} W$, as claimed. □

In the sequel we will use another, more concrete description of $\mathrm{Gr}^{\mathrm{rat}}$ first introduced in [42] (see also [20]). Let's start again from the linear space $\mathcal{P}$ of polynomials with complex coefficients; its (algebraic) dual $\mathcal{P}^*$ is simply the linear space $\mathbb{C}^\omega$ of countable sequences of complex numbers, with the pairing defined as follows: given $p = \{p_n\}_{n\in\mathbb{N}} \in \mathcal{P}$ and $\varphi = \{\varphi_n\}_{n\in\mathbb{N}} \in \mathcal{P}^*$, we have

$$\langle \varphi, p\rangle = \sum_{n\in\mathbb{N}} \varphi_n p_n$$

(notice that the sum is finite precisely because p is a polynomial). Consider now the family of linear functionals $\mathcal{E} = \{\mathrm{ev}_{r,\lambda}\}_{r\in\mathbb{N},\,\lambda\in\mathbb{C}}$ defined as follows:

$$\langle \mathrm{ev}_{r,\lambda}, p\rangle = p^{(r)}(\lambda)$$

where $p^{(r)}$ is the r-th derivative of p. It easy tho check that $\mathcal{E}$ is linearly independent set; we denote by $\mathscr{C}$ the linear subspace of $\mathcal{P}^*$ generated by $\mathcal{E}$. We think of $\mathscr{C}$ as a "space of differential conditions" we can impose on polynomials.

For every $\lambda \in \mathbb{C}$ we put

$$\mathscr{C}_\lambda := \operatorname{span}\{\mathrm{ev}_{r,\lambda}\}_{r\in\mathbb{N}}$$

Clearly $\mathscr{C} = \bigoplus_{\lambda\in\mathbb{C}} \mathscr{C}_\lambda$; moreover, for every $r \in \mathbb{N}$ we define

$$\mathscr{C}_\lambda^r := \operatorname{span}\{\mathrm{ev}_{s,\lambda}\}_{0\le s<r}$$

with the convention $\mathscr{C}_\lambda^0 = \{0\}$. Notice that the functionals

$$\frac{1}{r!}\,\mathrm{ev}_{r,0} \tag{1.30}$$

can be identified with the extraction of the r-th coefficient of a polynomial.

Definition 1.31. For every $c \in \mathscr{C}$ the (finite) set of points $\lambda \in \mathbb{C}$ such that the projection of c on $\mathscr{C}_\lambda$ is nonzero will be called the **support** of c.

Now let C be a finite-dimensional subspace in $\mathscr{C}$; we denote its annihilator in $\mathcal{P}$ by

$$V_C := \{\, p \in \mathcal{P} \mid \langle c, p\rangle = 0 \text{ for all } c \in C \,\}$$

Similarly, to every subspace $V \subseteq \mathcal{P}$ we associate its annihilator in $\mathscr{C}$:

$$\operatorname{Ann} V := \{\, c \in \mathscr{C} \mid \langle c, p\rangle = 0 \text{ for all } p \in V \,\}$$

Theorem 1.32. *A subspace $W \subseteq \mathcal{R}$ belongs to $\overline{\mathrm{Gr}}^{\mathrm{rat}}$ if and only if there exists a finite-dimensional subspace $C \subseteq \mathscr{C}$ and a polynomial q such that $W = q^{-1}V_C$; moreover $W \in \mathrm{Gr}^{\mathrm{rat}}$ if and only if $\deg q = \dim C$.*

Proof. Given C and q we put $W := q^{-1}V_C$. Let $\{\lambda_1, \dots, \lambda_s\}$ be the support of C; for every $i \in \{1, \dots, s\}$ let r_i be the maximum order of derivation of the functionals $\mathrm{ev}_{r,\lambda_i}$ involved in the elements of C. We define $p := \prod_{i=1}^{s}(z - \lambda_i)^{r_i+1}$; then every multiple of p belongs to V_C by construction, and in particular we have that $qp\mathcal{P} \subseteq V_C \subseteq \mathcal{P}$, or

$$p\mathcal{P} \subseteq W \subseteq \frac{1}{q}\mathcal{P}$$

This proves that $W \in \overline{\mathrm{Gr}}^{\mathrm{rat}}$. Now suppose that $\deg q = \dim C$, then by definition we have $\mathrm{codim}_{\mathcal{P}} V_C = \dim C = \deg q$ and theorem 1.29 tells us that $W \in \mathrm{Gr}^{\mathrm{rat}}$.

For the opposite implication take $W \in \overline{\mathrm{Gr}}^{\mathrm{rat}}$, then there exist $p, q \in \mathcal{P}$ such that $qp\mathcal{P} \subseteq qW \subseteq \mathcal{P}$; this means that the linear space qW can be defined by imposing a certain number of linearly independent conditions in the dual space of $\mathcal{P}/qp\mathcal{P}$. The latter can be identified with the space U of polynomials with degree less than $\deg p + \deg q$, so qW is determined by a finite-dimensional subspace in U^*. On the other hand, U^* is certainly generated by the elements of $\mathscr{C}$ (since, as we already noted, this space contains the functionals (1.30)). Thus we can single out a linear subspace $C \subseteq \mathscr{C}$ of finite dimension such that $V_C = qW$. If moreover $\mathrm{vdim}\, W = 0$ then by (1.28) the linear space $W' \cong qW/qp\mathcal{P}$ has dimension $\deg p$, so it is defined by $\deg p$ linearly independent conditions in U^*; it follows that the subspace C has dimension $\deg q$. □

We can assume without loss of generality that the polynomial q is monic. Denote by $\mathrm{Gr}_{\mathrm{fin}}\,\mathscr{C}$ the set of finite-dimensional linear subspaces of $\mathscr{C}$; the set

$$\mathrm{Gr}^{\mathrm{rat}*} := \{\, (C, q) \in \mathrm{Gr}_{\mathrm{fin}}\,\mathscr{C} \times \mathcal{P} \mid q \text{ monic of degree } \dim C \,\} \tag{1.33}$$

will be called the **dual rational Grassmannian**. Then theorem 1.32 defines a surjective map $\mathrm{Gr}^{\mathrm{rat}*} \to \mathrm{Gr}^{\mathrm{rat}}$, called the **dual map**, given by

$$(C, q)^* := q^{-1}V_C \tag{1.34}$$

This map is not injective (hence not a real pairing between dual spaces): for example every pair of the form $(\mathscr{C}_0^n, z^n)$ for some $n \in \mathbb{N}$ is mapped to $\mathcal{P}$.

The rôle of the polynomial q is clarified as follows. Let Γ_- be the multiplicative group of non-vanishing rational functions h defined on a disc centered at ∞ and such that $f(\infty) = 1$; using the embedding $\mathcal{R} \hookrightarrow \mathbb{C}((z^{-1}))$ defined by the Laurent series expansion at infinity (see Appendix B) such a function can be written as

$$h = 1 + \sum_{k \geq 1} h_k z^{-k} \tag{1.35}$$

This group acts on $\mathrm{Gr}^{\mathrm{rat}}$ in the obvious way ($h.W := hW$) and this action is free [39, Prop. 2.4].

Theorem 1.36. *The images by the dual map* (1.34) *of two points with the same conditions space C lie in the same Γ_--orbit of* $\mathrm{Gr}^{\mathrm{rat}}$.

Proof. Take (C, q_1) and $(C, q_2) \in \mathrm{Gr}^{\mathrm{rat}*}$ and let W_1 and W_2 be the corresponding points in $\mathrm{Gr}^{\mathrm{rat}}$. Then $\eta := \frac{q_1}{q_2}$ is the quotient of two monic polynomials of equal degree, i.e. an element of Γ_- and clearly

$$\eta W_1 = \frac{q_1}{q_2}\frac{1}{q_1}V_C = \frac{1}{q_2}V_C = W_2 \qquad \square$$

We will see later that the points of $\mathrm{Gr}^{\mathrm{rat}}$ that lie in the same Γ_--orbit give the same solution to the KP equation, so the choice of q only amounts to a sort of "gauge freedom".

1.2.2 1-point Grassmannians

Given $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$ we denote by $\mathrm{Gr}_{\lambda,k}$ the set of linear subspaces $W \in \mathrm{Gr}^{\mathrm{rat}}$ for which we can choose $p = q = (z-\lambda)^k$; in other words, we require that $\operatorname{vdim} W = 0$ and

$$(z-\lambda)^k \mathcal{P} \subseteq W \subseteq (z-\lambda)^{-k}\mathcal{P} \tag{1.37}$$

Then theorem 1.29 implies that the codimension of W in $(z-\lambda)^{-k}\mathcal{P}$ is k, whereas equation (1.28) implies that $\dim W' = k$ (with W' defined as usual by (1.24)), i.e. that the codimension of $(z-\lambda)^k\mathcal{P}$ in W is also k. Thus, each $W \in \mathrm{Gr}_{\lambda,k}$ is uniquely determined by the k-dimensional linear subspace W' in the $2k$-dimensional space

$$\frac{(z-\lambda)^{-k}\mathcal{P}}{(z-\lambda)^k\mathcal{P}} \tag{1.38}$$

i.e., each $\mathrm{Gr}_{\lambda,k}$ is isomorphic to the finite-dimensional Grassmannian $\mathrm{Gr}(k, 2k)$. We further define

$$\mathrm{Gr}_\lambda := \bigcup_{k\in\mathbb{N}} \mathrm{Gr}_{\lambda,k} \tag{1.39}$$

and call it the **1-point Grassmannian** at λ. From the point of view of the dual description of $\mathrm{Gr}^{\mathrm{rat}}$, elements of Gr_λ are simply the subspaces $W = (C, q)^*$ for which $C \subseteq \mathscr{C}_\lambda$, i.e. the subspaces determined by a set of conditions supported on a single point.

We remark that all 1-point Grassmannians are isomorphic: indeed the translation $z \mapsto z - \lambda$ gives a bijection between Gr_0 and Gr_λ for every $\lambda \in \mathbb{C}$. From now on we will usually consider the case $\lambda = 0$; in view of the above fact, any result we obtain will also hold for any other 1-point Grassmannian.

To gain a better understanding of the abstract structure of Gr_0 it is useful to reinterpret the definition (1.39) as an inductive limit. To this end consider again the family of finite-dimensional Grassmannians $\{\mathrm{Gr}_{0,k}\}_{k\geq 0}$; for each $k \in \mathbb{N}$ we can define a map $f_k \colon \mathrm{Gr}_{0,k} \to \mathrm{Gr}_{0,k+1}$ as follows. A subspace $W \in \mathrm{Gr}_{0,k}$ has the form $W = \operatorname{span}\{w_1, \dots, w_k\} \oplus z^k\mathcal{P}$; since we are only considering the linear structure, this can be equivalently written

$$W = \operatorname{span}\{w_1, \dots, w_k, z^k\} \oplus z^{k+1}\mathcal{P}$$

Thus we can see W as a subspace in $z^{-k-1}\mathcal{P} \subseteq \mathcal{R}$; quotienting by $z^{k+1}\mathcal{P}$ we obtain a $(k+1)$-dimensional linear subspace in $z^{-k-1}\mathcal{P}/z^{k+1}\mathcal{P}$, i.e. an element of $\mathrm{Gr}_{0,k+1}$. This defines an inductive system of sets $\{\mathrm{Gr}_{0,k}\}_{k\geq 0}$ and maps $\{f_k\}_{k\geq 0}$, and the corresponding limit is again Gr_0:

$$\varinjlim_{k\in\mathbb{N}} \mathrm{Gr}_{0,k} = \mathrm{Gr}_0$$

as it is immediate to verify.

We now describe how the well-known cell structure of Gr_0 (cfr. [37]) arises in this setting. In every linear space $z^{-k}\mathcal{P}/z^k\mathcal{P}$ we have the canonical basis

$$(z^{k-1}, z^{k-2}, \ldots, z, 1, z^{-1}, \ldots, z^{-k+1}, z^{-k})$$

This basis defines a complete flag

$$V_i := \mathrm{span}\{z^{k-1}, \ldots, z^{k-i}\} \quad \text{for all } 0 \leq i \leq 2k$$

Associated to this flag we have the corresponding decomposition of $\mathrm{Gr}_{0,k} \cong \mathrm{Gr}(k, 2k)$ in Schubert cells (cfr. Appendix C): for any $d \in \{0, \ldots, 2k\}$ the d-dimensional cells are indexed by the partitions of d of length not higher than k and whose parts are not greater than k, or in other words by the partitions whose Young diagram is contained in a square of side k. By passing to the limit $k \to \infty$, every partition is eventually included; so a point in Gr_0 is totally described by a partition p and a point $\vec{\alpha}$ in a (complex) affine cell of dimension $|p|$.

To recover the linear subspace of $\mathcal{R}$ associated to this data we proceed as follows: given p, let k be the least natural number such that the Young diagram of p is contained in a square of side k. Then select the corresponding Schubert cell C_σ in $\mathrm{Gr}(k, 2k)$ and pick the point $\vec{\alpha} \in C_\sigma$; this gives a subspace $W' \subseteq \frac{z^{-k}\mathcal{P}}{z^k\mathcal{P}}$, and the subspace of $\mathcal{R}$ we are looking for is $W = W' \oplus z^k\mathcal{P}$.

1.2.3 The adelic Grassmannian

The linear subspace $\mathcal{P} \subseteq \mathcal{R}$ is a very special one in that it belongs to every 1-point Grassmannian Gr_λ (indeed it is the only subspace that satisfies condition (1.37) for $k = 0$). Then it makes sense to give the following:

Definition 1.40. The **abstract adelic Grassmannian** is the restricted product

$$\mathrm{Gr}^{\mathrm{Ad}} := \prod_{\lambda\in\mathbb{C}}{}' \mathrm{Gr}_\lambda \tag{1.41}$$

with origin $\mathcal{P}$.

Here "restricted product with origin $\mathcal{P}$" means that a family of subspaces $\mathcal{W} = \{W_\lambda\}_{\lambda\in\mathbb{C}}$ belongs to $\mathrm{Gr}^{\mathrm{Ad}}$ if and only if $W_\lambda = \mathcal{P}$ for all but a *finite* set of points in $\mathbb{C}$; this finite set is called the **support** of $\mathcal{W}$. (Incidentally, the similarity between the previous definition and the construction of the adele ring of a global field explains the terminology "adelic".)

The description of 1-point Grassmannians as cell complexes obtained in the previous subsection immediately generalizes to $\mathrm{Gr}^{\mathrm{Ad}}$; indeed, a point $\mathcal{W} \in \mathrm{Gr}^{\mathrm{Ad}}$ is totally described by the following pieces of data:

- a finite set of points $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{C}$ (its support);
- for every λ_i a partition p_i and a point α_i in a complex cell of dimension $|p_i|$.

The points $\{\alpha_1, \ldots, \alpha_n\}$ (for a given support) will be called the **abstract Grassmannian coordinates** of $\mathcal{W}$.

We now show how the abstract adelic Grassmannian (1.41) can be embedded in the rational Grassmannian $\mathrm{Gr}^{\mathrm{rat}}$, following (and elaborating a bit on) the approach in [43, subsection 2.2]; this will also clarify the relationship between the dual mapping of subsection 1.2.1 and the abstract Grassmannian coordinates defined above.

For any $\lambda \in \mathbb{C}P^1$ and for any $f, g \in \mathcal{R}$ consider the symmetric bilinear form

$$\langle f, g\rangle_\lambda := \operatorname*{res}_{z=\lambda} f(z)g(z)\mathrm{d}z \tag{1.42}$$

and denote by $\mathrm{Ann}_\lambda W$ the annihilator of a subspace $W \subseteq \mathcal{R}$:

$$\mathrm{Ann}_\lambda(W) := \{\, f \in \mathcal{R} \mid \langle f, g\rangle_\lambda = 0 \text{ for every } g \in W \,\} \tag{1.43}$$

We will be particularly interested in the case $\lambda = \infty$, so let's try to get an explicit expression for $\langle f, g\rangle_\infty$. To this end we use again the embedding $\mathcal{R} \hookrightarrow \mathbb{C}((z^{-1}))$: let a and b be the order of (the Laurent series at infinity representing) f and g, respectively; then

$$\langle f, g\rangle_\infty = [z^{-1}] \sum_{i\le a} f_i z^i \sum_{j \le b} g_j z^j = [z^{-1}] \sum_{k \le a+b} \left(\sum_{\ell=0}^{a+b-k} f_{a-\ell} g_{k+\ell-a} \right) z^k$$

so that

$$\langle f, g\rangle_\infty = \sum_{k=0}^{a+b+1} f_{a-k} g_{k-a-1}$$

and in particular

$$\langle z^a, z^b\rangle_\infty = \sum_{k=0}^{a+b+1} \delta_{k,0}\delta_{k,a+b+1} = \begin{cases} 1 & \text{if } a = -b-1 \\ 0 & \text{otherwise} \end{cases}$$

It follows immediately that $\langle \mathcal{P}, \mathcal{P}\rangle_\infty = 0$ and $\langle \mathcal{R}_-, \mathcal{R}_-\rangle_\infty = 0$, i.e. $\mathcal{P}$ and $\mathcal{R}_-$ are two (evidently maximal) isotropic subspaces for the inner product space $(\mathcal{R}, \langle \,\cdot\,, \,\cdot\, \rangle_\infty)$.

Given a subspace $W \subseteq \mathcal{R}$ we define

$$W^* := \mathrm{Ann}_\infty W = \{\, f \in \mathcal{R} \mid \langle f, g\rangle_\infty = 0 \text{ for all } g \in W \,\} \tag{1.44}$$

Now consider a subspace of the form hW for some $h \in \mathcal{R}$; we claim that

$$(hW)^* = h^{-1} W^* \tag{1.45}$$

Indeed, using (1.44),

$$(hW)^* = \{ f \in \mathcal{R} \mid \langle f, hg \rangle_\infty = 0 \text{ for all } g \in W \}$$

but $\langle f, hg \rangle_\infty = \langle hf, g \rangle_\infty$, hence $f \in (hW)^*$ if and only if $hf \in W^*$, as claimed. From this we deduce that the correspondence $W \mapsto W^*$ defines an involution on $\overline{\mathrm{Gr}}^{\mathrm{rat}}$: indeed if $W \in \overline{\mathrm{Gr}}^{\mathrm{rat}}$ then there exists $p \in \mathcal{P}$ such that $p\mathcal{P} \subseteq W \subseteq p^{-1}\mathcal{P}$; taking annihilators and using (1.45) it follows that $p\mathcal{P} \subseteq W^* \subseteq p^{-1}\mathcal{P}$ and so W^* also belongs to $\overline{\mathrm{Gr}}^{\mathrm{rat}}$. The map $W \to W^*$ is called the **adjoint involution**; notice that it preserves $\mathrm{Gr}^{\mathrm{rat}}$ and each Gr_λ.

Now we can finally define the embedding $i\colon\ \mathrm{Gr}^{\mathrm{Ad}} \to \mathrm{Gr}^{\mathrm{rat}}$ with the following recipe: given $\mathcal{W} = \{W_\lambda\}_{\lambda \in \mathbb{C}} \in \mathrm{Gr}^{\mathrm{Ad}}$ we put

$$\mathcal{W} \mapsto W := \bigcap_{\lambda \in \mathbb{C}} \operatorname{Ann}_\lambda W_\lambda^* \tag{1.46}$$

More concretely, the embedding goes as follows. When $W_\lambda = \mathcal{P}$ then $\operatorname{Ann}_\lambda \mathcal{P}^* = \operatorname{Ann}_\lambda \mathcal{P}$ is just the set of functions regular in λ, so W contains rational functions which are regular everywhere on $\mathbb{C}P^1$ except at infinity and at the support Λ of $\mathcal{W}$. Moreover the residue theorem applied to a small circle around infinity tells us that

$$\langle f, g \rangle_\infty = \operatorname*{res}_{z=\infty} f(z)g(z)\mathrm{d}z = -\sum_{\lambda \in \mathbb{C}} \operatorname*{res}_{z=\lambda} f(z)g(z)\mathrm{d}z = -\sum_{\lambda \in \mathbb{C}} \langle f, g \rangle_\lambda \tag{1.47}$$

Now if the support Λ of $\mathcal{W}$ consists of a single point λ then, by the previous equality, we have $\operatorname{Ann}_\lambda V = \operatorname{Ann}_\infty V = V^*$ for any subspace V, and in particular $\operatorname{Ann}_\lambda W_\lambda^* = (W_\lambda^*)^* = W_\lambda$; hence in this case $i(\mathcal{W}) = W_\lambda$. This of course coincides with the trivial embedding of Gr_λ considered as a subset of $\mathrm{Gr}^{\mathrm{rat}}$.

In the general case, choose for each $\lambda \in \Lambda$ a natural number k_λ such that $W_\lambda^* \in \mathrm{Gr}_{\lambda, k_\lambda}$; then there exists a basis for W_λ^* of the form

$$\{ \omega_1, \dots, \omega_{k_\lambda}, (z-\lambda)^{k_\lambda}, (z-\lambda)^{k_\lambda+1}, \dots \}$$

where the ω_i are Laurent polynomials in $z - \lambda$. Then $\operatorname{Ann}_\lambda W_\lambda^*$ consists of all rational functions f that have a pole of order at most k_λ at λ and that satisfy the k_λ conditions

$$\langle f, \omega_i \rangle_\lambda = 0 \qquad \text{for all } 1 \le i \le k_\lambda \tag{1.48}$$

Each such equation amounts to a homogeneous linear condition on the coefficients of the Laurent series of f at λ, i.e. to an element of $\mathscr{C}_\lambda$.

So for each $\lambda \in \Lambda$ we describe W_λ as a space of functions which are regular except for a pole at ∞ and a pole of order at most k_λ at λ, and satisfying the k_λ conditions (1.48); then $i(\mathcal{W})$ is simply the linear subspace of $\mathcal{R}$ obtained by imposing simultaneously all these conditions at the various points $\lambda \in \Lambda$.

We denote by $\mathrm{Gr}^{\mathrm{ad}}$ the image of the embedding i in $\mathrm{Gr}^{\mathrm{rat}}$. The relationship with the dual description for subspaces in $\mathrm{Gr}^{\mathrm{rat}}$ is as follows: let's call a space of conditions $C \subseteq \mathscr{C}$

homogeneous if it admits a basis made by "1-point conditions", i.e. differential conditions each one involving a single point:

$$C = \bigoplus_{\lambda} (C \cap \mathscr{C}_\lambda) \tag{1.49}$$

Notice that for every homogeneous subspace C we can canonically build a polynomial of degree $\dim C$ in the following manner:

$$q_C := \prod_{\lambda \in \mathbb{C}} (z - \lambda)^{n_\lambda} \tag{1.50}$$

where $n_\lambda := \dim(C \cap \mathscr{C}_\lambda)$. Then $\mathrm{Gr}^{\mathrm{ad}}$ is simply the image with respect to the dual map (1.34) of the homogeneous finite-dimensional subspaces of $\mathscr{C}$, with the polynomial q as defined above:

$$\mathrm{Gr}^{\mathrm{ad}} = \{ W \in \mathrm{Gr}^{\mathrm{rat}} \mid W = (C, q)^* \text{ with } C \text{ homogeneous and } q = q_C \}$$

This gives an independent description of $\mathrm{Gr}^{\mathrm{ad}}$ inside $\mathrm{Gr}^{\mathrm{rat}}$ first obtained in [42].

1.2.4 The Segal-Wilson Grassmannian

This Grassmannian was introduced in [39] (see also [37, Ch. 7] for a slightly different version). Consider the separable complex Hilbert space $L^2(S^1, \mathbb{C})$ of square-integrable functions on the circle. We can view its elements as functions $\mathbb{C} \to \mathbb{C}$ by embedding S^1 in the complex plane as the circle γ_R with center 0 and radius $R \in \mathbb{R}^+$; let's call $H(R)$ the Hilbert space so obtained. Every such space comes equipped with the orthonormal basis $\{z^k\}_{k\in\mathbb{Z}}$ via the Laurent expansion of $f \in H(R)$ in an annulus centered at the origin and containing γ_R. We further define $H_+(R) := \mathrm{span}\{z^k\}_{k\geq 0}$ and $H_-(R) := \mathrm{span}\{z^k\}_{k<0}$ (so that $H(R) = H_+(R) \oplus H_-(R)$) and denote by $\pi_\pm$ the orthogonal projections on $H_\pm(R)$. Notice that $H_+(R)$ may be seen as the space of functions $\gamma_R \to \mathbb{C}$ that extends to holomorphic functions on the disc $D_0(R) := \{ z \in \mathbb{C}P^1 \mid |z| \leq R \}$, and similarly $H_-(R)$ may be seen as the space of functions $\gamma_R \to \mathbb{C}$ that extends to holomorphic functions on the disc $D_\infty(R) := \{z \in \mathbb{C}P^1 | \, |z| \geq R\}$ and vanishing at infinity.

The **full Segal-Wilson Grassmannian** of $H(R)$, denoted $\overline{\mathrm{Gr}}(R)$, is the set of all closed linear subspaces $W \subseteq H(R)$ such that $\pi_+|_W$ is a Fredholm operator and $\pi_-|_W$ is a compact operator. One can prove (cfr. [37]) that $\overline{\mathrm{Gr}}(R)$ is a Banach manifold modeled on the space of compact operators $H_+(R) \to H_-(R)$, and that it decomposes in a countable number of connected components labeled by the index of π_+. Since every component is isomorphic to each other (via multiplication by z^k, with $k \in \mathbb{Z}$) it suffices to consider one of them; we take the one consisting of subspaces of virtual dimension zero (i.e., the one that contains H_+) and call it the **Segal-Wilson Grassmannian** of $H(R)$, denoted $\mathrm{Gr}(R)$.

The relationship between the Segal-Wilson Grassmannian and the rational Grassmannian previously introduced is worked out in [43, subsection 2.5]. Very briefly, the story goes as follows: for every $W \in \overline{\mathrm{Gr}}^{\mathrm{rat}}$ we can choose $R \in \mathbb{R}^+$ such that every root of q (the polynomial that appear in the condition (1.23)) is contained in the open disc $|z| < R$; then for every $f \in W$ the restriction $f|_{\gamma_R}$ is continuous, hence square-integrable on γ_R; so W determines a linear subspace in $H(R)$ whose L^2-closure belongs to $\overline{\mathrm{Gr}}(R)$[5]. By considering only the virtual dimension zero subspaces we get the corresponding embedding of $\mathrm{Gr}^{\mathrm{rat}}$ in $\mathrm{Gr}(R)$.

We can use this embedding to define a topology on $\overline{\mathrm{Gr}}^{\mathrm{rat}}$ (and hence on its subspaces, including $\mathrm{Gr}^{\mathrm{ad}}$) by restriction of the topology naturally defined on each $\mathrm{Gr}(R)$.

Now let $\Gamma_+(R)$ be the group of non-vanishing analytic functions $g\colon \gamma_R \to \mathbb{C}^*$ that extend to holomorphic functions on the disc $D_0(R)$ and such that $g(0) = 1$; this group acts from the right on $\mathrm{Gr}(R)$ by $W \mapsto Wg^{-1}$. By the very definition, every $g \in \Gamma_+(R)$ can be written as

$$g(z) = 1 + \sum_{k \geq 1} h_k z^k \tag{1.51}$$

where the series has radius of convergence greater than R, so that g is completely determined by the family of coefficients $\{h_k\}_{k\geq 1}$. Alternatively, since g is defined on a simply connected domain there surely exists a holomorphic function f such that $g = \mathrm{e}^f$, and expanding f in a power series (again with radius of convergence greater than R) we can write

$$g(z) = \exp \sum_{i \geq 1} t_i z^i \tag{1.52}$$

so that g is also determined by the family of coefficients $\{t_k\}_{k\geq 1}$ (for the relationship between these two sets of coefficients see Appendix A). From the latter representation we see that the group we called Γ can be embedded in each $\Gamma_+(R)$ by mapping $\boldsymbol{t} \in \Gamma$ to the expression (1.52), i.e. to the entire function e^p where p is the polynomial with coefficients $\boldsymbol{t}$. We define an action of Γ on each $\mathrm{Gr}(R)$ (hence on $\mathrm{Gr}^{\mathrm{rat}}$) as follows:

$$\boldsymbol{t}.W := W\mathrm{e}^{-p} \tag{1.53}$$

We observe that this action preserves each Gr_λ and so preserves $\mathrm{Gr}^{\mathrm{ad}}$; we conclude that the abelian group Γ acts continuously on the adelic Grassmannian, and we will see that this action corresponds to the KP flows on the space of rational solutions.

1.3 The KP hierarchy

In this Section we define the KP hierarchy of integrable partial differential equations. For the general background and the omitted proofs we refer the reader to [2, 12].

[5]More precisely, it belongs to the sub-Grassmannian denoted "Gr_1" in [39] (notice that R is fixed to 1 in that paper); and vice versa, every element of this sub-Grassmannian corresponds to an element of $\overline{\mathrm{Gr}}^{\mathrm{rat}}$ for which we can take $R = 1$.

1.3.1 Pseudo-differential operators

Let $\mathcal{A}$ be a differential algebra, i.e. a unitary and associative algebra equipped with a *derivation* D, a map $D\colon \mathcal{A} \to \mathcal{A}$ that satisfies Leibniz's law (namely $D(ab) = D(a)b + aD(b)$). The *algebra of pseudo-differential operators over* $\mathcal{A}$, denoted $\Psi(\mathcal{A})$, is defined as follows. As a linear space, it is just the space of formal Laurent series with coefficients in $\mathcal{A}$ with respect to a formal variable that we call D^{-1}; so every nonzero element of $\Psi(\mathcal{A})$ can be written in the canonical form

$$Q = \sum_{i \le a} q_i D^i \qquad \text{for some } a \in \mathbb{Z} \tag{1.54}$$

with $q_i \in \mathcal{A}$ and $q_a \neq 0$; we say that Q has **order** a and write $\operatorname{ord} Q = a$. The sum of two series and the multiplication of a series by an element of $\mathcal{A}$ are defined in the usual way. Looking at the canonical form (1.54) we see that to multiply two such series it is sufficient to know the commutation rule between D^i (for all $i \in \mathbb{Z}$) and a generic element of $\mathcal{A}$. Now, when $i = 1$ Leibniz's rule tells us that

$$Df = fD + f'$$

and this is immediately generalized by induction to every $i \ge 0$:

$$D^i f = \sum_{k \ge 0} \binom{i}{k} f^{(k)} D^{i-k} \tag{1.55}$$

But in the present setting this expression makes sense also for $i < 0$, because it gives precisely a formal power series in D^{-1} with coefficients in $\mathcal{A}$; for example when $i = -1$ we obtain[6]

$$D^{-1} f = \sum_{k \ge 0} (-1)^k f^{(k)} D^{-1-k} = fD^{-1} - f'D^{-2} + f''D^{-3} + \dots$$

So the relation (1.55) completely determines the product in $\Psi(\mathcal{A})$, making it an associative algebra.

On $\Psi(\mathcal{A})$ we have the projection operators

$$Q \mapsto Q_- := \sum_{i \le -1} q_i D^i \quad \text{and} \quad Q \mapsto Q_+ := \sum_{0 \le i \le a} q_i D^i$$

We denote by $\Psi_\pm(\mathcal{A})$ their images; note that $\Psi_+(\mathcal{A})$ is nothing but the algebra of "genuine" polynomial differential operators with coefficients in $\mathcal{A}$, whereas the elements of $\Psi_-(\mathcal{A})$ are formal integral operators; we will call them (formal) **Volterra operators** and $\Psi_-(\mathcal{A})$ the **Volterra (sub)algebra**.

[6]Recall that for every $n > 0$ we can define $\binom{-n}{k} := (-1)^k \binom{n+k-1}{k}$.

1.3.2 The KP hierarchy

From now on we assume that the elements of our differential algebra $\mathcal{A}$ are (smooth) functions of a countable set of variables $\boldsymbol{t} := \{t_i\}_{i\geq 1}$; these variables will serve as the times of the hierarchy. As we will see, the variable x whose D is the derivation operator can be identified with t_1.

There are several equivalent ways to introduce the KP hierarchy; here we briefly summarize the three approaches that will be used in this work.

1. Let ϕ be a pseudo-differential operator of the form

$$\phi = 1 + \sum_{i\geq 1} a_i D^{-i} \tag{1.56}$$

These operators form a group G under multiplication. For every $k \geq 1$ we define the k-th KP flow on G by the following equation ("Sato form of the KP hierarchy"):

$$\partial_k \phi = -(\phi D^k \phi^{-1})_- \phi \tag{1.57}$$

where $\partial_k := \frac{\partial}{\partial t_k}$ is understood to act on the coefficients of the power series (1.56). Notice that (1.57) is an equality between Volterra operators; by comparing the two sides at each order in D^{-1} we get an infinite family of partial differential equations in the unknowns $\{a_i\}_{i\geq 1}$.

2. Let Q be a pseudo-differential operator of the form

$$Q = D + \sum_{i\geq 1} q_i D^{-i} \tag{1.58}$$

and denote by $\mathcal{Q}$ the corresponding subset of $\Psi(\mathcal{A})$. The k-th KP flow on $\mathcal{Q}$ is defined by the following equation ("Lax form of the KP hierarchy")[7]

$$\partial_k Q = [Q^k_+, Q] \tag{1.59}$$

We claim that this is again an equality between Volterra operators. Indeed on the left-hand side every non-negative power of D has constant coefficient and so gets annihilated by ∂_k, whereas the right-hand side can be rewritten as

$$[Q^k_+, Q] = -[Q^k_-, Q] \tag{1.60}$$

because $Q^k_+ = Q^k - Q^k_-$ and Q^k obviously commutes with Q. But the latter commutator has (maximum) order $-1+1-1=-1$, thus it belongs to $\Psi_-(\mathcal{A})$. We conclude that equation (1.59), just as (1.57), yields another infinite family of partial differential equations, this time in the unknown functions $\{q_i\}_{i\geq 1}$.

[7] Here and everywhere in the sequel we use the shorthand notation $Q^k_\pm := (Q^k)_\pm$.

3. Finally, we can view the KP hierarchy as the compatibility conditions for the linear system
$$Q\psi = z\psi \qquad \partial_k \psi = Q_+^k \psi \tag{1.61}$$
where $Q \in \mathcal{Q}$ and ψ is the so-called **formal Baker function** (or **wave function**), which is an expression of the form
$$\psi(\boldsymbol{t}, z) = \Big(1 + \sum_{i\geq 1} a_i(\boldsymbol{t}) z^{-i}\Big) \mathrm{e}^{\xi(\boldsymbol{t},z)} \tag{1.62}$$
where we have defined
$$\xi(\boldsymbol{t}, z) := \sum_{i\geq 1} t_i z^i \tag{1.63}$$

The relationship between the objects ϕ, Q and ψ is as follows. The two pseudo-differential operator are related by the equation
$$Q = \phi D \phi^{-1} \tag{1.64}$$
a procedure known in the literature as **dressing** [12]. The two sets of coefficients $\{a_i\}$ and $\{q_i\}$ are related by a system of equations of the form
$$a_i' + q_i + H_i(q, a) = 0 \qquad (i \geq 1) \tag{1.65}$$
where each H_i is a differential polynomial that only depends on $\{a_1, \ldots, a_{i-1}\}$ (with derivatives up to order $i-2$) and $\{q_1, \ldots, q_{i-1}\}$ (undifferentiated); for example
$$q_1 = -a_1' \tag{1.65a}$$
$$q_2 = -a_2' - q_1 a_1 \tag{1.65b}$$
$$q_3 = -a_3' - q_2 a_1 + q_1 a_1' - q_1 a_2 \tag{1.65c}$$
and so on. Notice that the correspondence is not one-to-one: if Q is given, the operator ϕ satisfying (1.64) is determined only up to transformations of the form
$$\phi \mapsto \phi(1 + C) \tag{1.66}$$
where $C \in \Psi_-(\mathcal{A})$ has constant coefficients.

The formal Baker function may be seen as a "commutative version" of ϕ: indeed the two objects are related by[8]
$$\psi(\boldsymbol{t}, z) = \phi.\mathrm{e}^{\xi(\boldsymbol{t},z)} \tag{1.67}$$
where the action of a pseudo-differential operator on the formal exponential $\mathrm{e}^{\xi(\boldsymbol{t},z)}$ is defined by the equality $D.\mathrm{e}^{\xi(\boldsymbol{t},z)} := z\mathrm{e}^{\xi(\boldsymbol{t},z)}$. It follows in particular that if ϕ is given by (1.56), then ψ is given by (1.62) where the functions $\{a_i\}_{i\geq 1}$ are exactly the same, so that the correspondence between ϕ and ψ is one-to-one.

[8] More precisely, formal Baker functions live in a free rank 1 module over $\Psi(\mathcal{A})$ with generator $\mathrm{e}^{\xi(\boldsymbol{t},z)}$, see [39, Sect. 4].

To explain the name "KP hierarchy" for the dynamical system defined above we write down explicitly some of the simplest cases of equation (1.59). For $k = 1$ we have

$$\partial_1 Q = [Q_+, Q] = [D, Q] = \sum_{i \geq 1} q_i' D^{-i} \tag{1.68}$$

whence a system of the form

$$\partial_1 q_i = q_i' \qquad \text{for all } i \geq 1$$

This simply means, as anticipated, that the first time variable t_1 may be identified with x. For $k = 2$ we have $Q_+^2 = D^2 + 2q_1$ and the first equations of the corresponding hierarchy are

$$\frac{\partial q_1}{\partial t_2} = q_1'' + 2q_2' \qquad \frac{\partial q_2}{\partial t_2} = q_2'' + 2q_3' + 2q_1' q_1 \qquad \text{etc.} \tag{1.69}$$

For $k = 3$ we have $Q_+^3 = D^3 + 3q_1 D + 3(q_1' + q_2)$, whence

$$\frac{\partial q_1}{\partial t_3} = q_1''' + 3q_2'' + 3q_3' + 6q_1' q_1 \qquad \text{etc.} \tag{1.70}$$

Now consider the system consisting of the three equations above:

$$\begin{cases} \partial_2 q_1 = q_1'' + 2q_2' \\ \partial_2 q_2 = q_2'' + 2q_3' + 2q_1' q_1 \\ \partial_3 q_1 = q_1''' + 3q_2'' + 3q_3' + 6q_1' q_1 \end{cases}$$

If we substitute for q_3' and q_2 (by deriving in a suitable manner) we get the single equation

$$3\partial_2^2 q_1 - 4\partial_3 q_1' + (q_1''' + 12q_1' q_1)' = 0$$

and putting $t_2 = y$, $t_3 = t$ and $u = 2q_1$ we can rewrite it as

$$\frac{3}{4} u_{yy} = (u_t - \frac{1}{4} u_{xxx} - \frac{3}{2} u_x u)_x \tag{1.71}$$

This is exactly equation (1) in the Introduction, i.e. the Kadomtsev-Petviashvili equation; in this sense this PDE is the first nontrivial member of the hierarchy of equations (1.59).

1.3.3 Solutions to the KP hierarchy coming from the Segal-Wilson Grassmannian

Here we briefly review the relationship discovered by Sato [38] between solutions of the KP hierarchy and points of infinite-dimensional Grassmannians, using the very general framework developed by Segal and Wilson in [39].

In the following Gr and Γ_+ will denote $\mathrm{Gr}(R)$ and $\Gamma_+(R)$ for some fixed choice of R. For every $W \in \mathrm{Gr}$ we define

$$\Gamma_+^W := \{ g \in \Gamma_+ \mid W g^{-1} \text{ is transverse} \} \tag{1.72}$$

(a subspace $W \in \mathrm{Gr}$ is called transverse if $\pi_+|_W$ is an isomorphism). For every $g \in \Gamma_+^W$ we consider the inverse image of $1 \in H_+$ by π_+, which is a function $\tilde{\psi}_W \in Wg^{-1}$ called the **reduced Baker function** associated to W and g. Since $\pi_+(\tilde{\psi}_W) = 1$ it cannot contain positive powers of z, so it has the form

$$\tilde{\psi}_W = 1 + \sum_{i \geq 1} a_i(g) z^{-i} \tag{1.73}$$

for some coefficients a_i (that turns out to be meromorphic functions on Γ_+). If we now multiply $\tilde{\psi}_W$ by the function g we get an element in the original subspace W; the **Baker function** associated to W is the map ψ_W which sends $g \in \Gamma_+^W$ to this element:

$$\psi_W(g, z) = \left(1 + \sum_{i \geq 1} a_i(g) z^{-i}\right) g(z) \tag{1.74}$$

We also define the **stationary Baker function** as the restriction of ψ_W to the 1-parameter subgroup $\{\mathrm{e}^{xz}\}_{x \in \mathbb{C}}$: $\psi_W(x, z) := \psi_W(e^{xz}, z)$.

Now we write $g \in \Gamma_+^W$ according to the representation (1.52); then equation (1.74) reads

$$\psi_W(\boldsymbol{t}, z) = \left(1 + \sum_{i \geq 1} a_i(\boldsymbol{t}) z^{-i}\right) \mathrm{e}^{\xi(\boldsymbol{t}, z)} \tag{1.75}$$

which is an expression exactly analogous to (1.62), and so corresponds to the following element of the group G:

$$\phi_W := 1 + \sum_{i \geq 1} a_i(\boldsymbol{t}) D^{-i} \tag{1.76}$$

Theorem 1.77. *The operator defined by* (1.76) *satisfies the Sato form of the KP equation* (1.57).

This fact was first proven in [39]. To obtain the corresponding solution for the KP hierarchy in Lax form (1.59) we need the associated order one operator $Q_W = \phi_W D \phi_W^{-1}$; recall that this involves a many-to-one correspondence, so that the points of Gr which give operators (1.76) related by a transformation of the form (1.66) all determine the same solution. But it easy to see that such points are precisely the members of a Γ_--orbit in Gr; hence the action of Γ_- does not alter the associated solution, as anticipated.

We remark that the infinite times t_k of the hierarchy are identified with the coefficients $\boldsymbol{t}$ determining $g \in \Gamma_+$; in particular the group of finite combinations of KP flows is identified with $\Gamma \subset \Gamma_+$.

1.3.4 The tau function

Another way to describe the solutions to the KP hierarchy involves the so-called **tau function**; the theory of this object is also developed in [39]. For our purposes, it suffices to say that for each $W \in \mathrm{Gr}(R)$ there exists a holomorphic function τ_W defined up to constant factors on $\Gamma_+(R)$ and such that the following equality holds ("Sato's formula"):

$$\tilde{\psi}_W(g,z) = \frac{\tau_W(gq_z)}{\tau_W(g)} \tag{1.78}$$

where for each $z \in \mathbb{C}$ we define $q_z \colon \mathbb{C}P^1 \to \mathbb{C}P^1$ as $q_z(\zeta) := 1 - \frac{\zeta}{z}$. Formula (1.78) should be interpreted as follows: at $g \in \Gamma_+^W(R)$ fixed, the reduced Baker function $z \mapsto \tilde{\psi}_W(g,z)$ clearly extends to the disc $D_\infty(R)$; then Sato's formula tells us that in the interior of that disc (where $q_z \in \Gamma_+(R)$ as a function of ζ) it coincides with the given ratio. Thus, by continuity, the same formula also holds on the circle γ_R.

If we now write g in the exponential representation (1.52) and expand q_z as

$$q_z(\zeta) = \exp\log(1 - \frac{\zeta}{z}) = \exp\Big(-\sum_{k\geq 1}\frac{\zeta^k}{kz^k}\Big) \tag{1.79}$$

then Sato's formula (1.78) becomes

$$\tilde{\psi}_W(\boldsymbol{t},z) = \frac{\tau_W(\boldsymbol{t} - [z^{-1}])}{\tau_W(\boldsymbol{t})} \tag{1.80}$$

where we have defined the "Miwa shift" on the coefficients $\boldsymbol{t}$ as

$$\boldsymbol{t} - [z^{-1}] := \{t_k - \frac{1}{kz^k}\}_{k\geq 1} = \{x - \frac{1}{z}, t_2 - \frac{1}{2z^2}, t_3 - \frac{1}{3z^3}, \dots\}$$

By expanding $\tau_W(\boldsymbol{t} - [z^{-1}])$ in a Taylor series around $z = \infty$ and substituting in (1.80) we can express the coefficients $a_i(g)$ of the Baker function associated to W in terms of the corresponding tau function τ_W; in general, they will have the form

$$a_i = -\frac{1}{i}\partial_i \log\tau_W + \frac{P_i(\tau_W)}{\tau_W}$$

where P_i is a polynomial with constant coefficients in the differential operators $\partial_1 \dots \partial_{i-1}$. In particular we have

$$a_1 = -\frac{\partial}{\partial x}\log\tau_W$$

and since the solution of the KP equation associated to ψ_W is $u = 2q_1 = -2a_1'$ we get the celebrated formula

$$u(\boldsymbol{t}) = 2\frac{\partial^2}{\partial x^2}\log\tau_W(\boldsymbol{t}) \tag{1.81}$$

for the general solution of the KP equation associated to a point $W \in \mathrm{Gr}$ in terms of its tau function.

In the sequel we will need an expression for translating the action of Γ_+ on Gr to the corresponding tau functions. This is given by the formula

$$\tau_{Wg^{-1}}(h) = \frac{\tau_W(gh)}{\tau_W(g)} = \tau_W(gh)$$

where the second equality follows by ignoring the constant (i.e., not depending on h) factor $\tau_W(g)^{-1}$. Using the variables $\boldsymbol{t}$ this becomes

$$\tau_{\boldsymbol{t}.W}(\boldsymbol{s}) = \tau_W(\boldsymbol{t}+\boldsymbol{s}) \tag{1.82}$$

It follows that, if we fix the normalization of τ in some way, the KP flows $\Gamma \subset \Gamma_+$ act on τ simply as a translation of the corresponding times.

1.3.5 Rationality of the solutions and the adelic Grassmannian

We are now interested in the solutions of the KP equation (or, more generally, of the whole KP hierarchy) such that the function u is a proper (i.e., vanishing at infinity) rational function of the variable x. Recall that $u = -2a_1'$, where a_1 is the coefficient of z^{-1} in the Baker function ψ associated to the solution. Suppose now that a_1 is a proper rational function of x. By [39, Prop. 5.17] we know that a_1 has at most simple poles, hence u has the form

$$u = 2\sum_{j=1}^{n} \frac{u_j(y,t)}{(x - x_j(y,t))^2}$$

Substituting this expression into the KP equation we see that it must be $u_j = -1$ for all j, so the solutions we are after have the form

$$u = -2\sum_{j=1}^{n} \frac{1}{(x - x_j(y,t))^2} \tag{1.83}$$

Equivalently, by (1.81), these solutions come from points $W \in \mathrm{Gr}^{\mathrm{rat}}$ whose tau function has the form

$$\tau(x,y,t) = \prod_{j=1}^{n} (x - x_j(y,t)) \tag{1.84}$$

More generally, the family of functions $\{q_i\}_{i\geq 1}$ are expressed by (1.65) in terms of differential polynomials of the $\{a_i\}_{i\geq 1}$, so to get rational solutions of the KP hierarchy we should look for the points $W \in \mathrm{Gr}^{\mathrm{rat}}$ such that all the a_i's are proper rational functions of x, or equivalently such that the reduced Baker function $\tilde{\psi}_W = 1 + \sum_i a_i z^{-i}$ is a rational function of x with limit 1 as $x \to \infty$. Such points are completely characterized by the following:

Theorem 1.85. *Let $W \in \mathrm{Gr}^{\mathrm{rat}}$. The following are equivalent:*
1) *$W \in \mathrm{Gr}^{\mathrm{ad}}$;*
2) *$\tilde{\psi}_W$ is a rational function of x that tends to 1 as $x \to \infty$;*
3) *τ_W is a polynomial in x with constant leading coefficient.*

The equivalence $1 \leftrightarrow 2$ is proven in [42] and the equivalence $2 \leftrightarrow 3$ is an easy consequence of Sato's formula (see also [20]). We can use property 3 to fix the normalization of the tau function by making τ_W a monic polynomial in x, as in (1.84).

To give an idea of why theorem 1.85 holds we summon up a pair of well-known formulas for the Baker function and the tau function of a point W in $\mathrm{Gr}^{\mathrm{rat}}$. These are most easily expressed using the dual description of subsection 1.2.1; so take $(C, q) \in \mathrm{Gr}^{\mathrm{rat}*}$ and let $W = (C, q)^*$ be the corresponding point in $\mathrm{Gr}^{\mathrm{rat}}$. Let $d := \dim C$; we claim that it suffices to consider the case $q = z^d$. Indeed for any other choice of q we obtain, by theorem 1.36, a subspace related to $\left(C, z^d\right)^*$ by a rational function $\eta \in \Gamma_-$, and the action of such a function on ψ and τ is known [39]: we have

$$\psi_{\eta W} = \psi_W \eta \qquad \text{and} \qquad \tau_{\eta W} = \hat{\eta}\tau_W \tag{1.86}$$

where the map $\eta \mapsto \hat{\eta}$ is defined by sending $\mathrm{e}^{c_k z^{-k}}$ to $\mathrm{e}^{-k c_k t_k}$ for every $k > 0$.

So we fix $q = z^d$ and take a basis $(c_1, \ldots, c_d)$ for C. In [42, Sect. 4] it is argued that the Baker function ψ_W must have the form

$$\psi_W(g, z) = (1 + a_1(g)z^{-1} + \cdots + a_d(g)z^{-d})g(z) \tag{1.87a}$$

where the a_i are determined by imposing the d conditions $\langle c_i, z^d \psi_W(g, z)\rangle = 0$ (for $i = 1 \ldots d$) identically in g. More explicitly, we have the linear system of equations

$$\langle c_i, (z^d + \sum_{j=1}^{d} a_j(g) z^{d-j}) g(z)\rangle = 0 \tag{1.87b}$$

in the unknowns $\{a_1, \ldots, a_d\}$, whose coefficients are expressions involving g and its derivatives evaluated at the points in the support of C.

The tau function associated to $W \in \mathrm{Gr}^{\mathrm{rat}}$ also admits a very neat formula that goes back at least to Krichever. Let again $(c_1, \ldots, c_d)$ be a basis for C with $q = z^d$; then

$$\tau_W(g) = \det(\langle c_i, z^{j-1} g\rangle)_{i,j=1\ldots d} \tag{1.88}$$

Notice that the matrix defined above is just the matrix of coefficients of the linear system (1.87b), so that the system is determined (and the Baker function exists) exactly when $\tau_W(g) \neq 0$, i.e. $g \in \Gamma_+^W$. We also remark that, since $\partial_x \mathrm{e}^{\xi(\boldsymbol{t},z)} = z\mathrm{e}^{\xi(\boldsymbol{t},z)}$, formula (1.88) may be written also as

$$\tau_W(g) = \det(\langle c_i, \partial_x^{j-1} g\rangle)_{i,j=1\ldots d} \tag{1.89}$$

i.e., as the Wronskian determinant of the functions $\langle c_i, g\rangle$. The choice of another basis for C only produces a multiplicative constant, which is irrelevant for tau functions.

Now let's return to theorem 1.85. Take a point $W \in \mathrm{Gr}^{\mathrm{ad}}$, then by the discussion at the end of subsection 1.2.3 we know that $W = (C, q_C)^*$ where C is a homogeneous space of conditions; let $(c_1, \ldots, c_d)$ be a basis of 1-point conditions for it. Now leave for a moment the adelic Grassmannian and consider the subspace $U := (C, z^d)^* \in \mathrm{Gr}^{\mathrm{rat}}$, so that we can apply the previous formulas. Since each c_i involves only a single point, from each equation in the system (1.87b) we can factor out and simplify a term $g(\lambda_i)$ and this leaves a system for the a_j's which has as coefficients power series such as $\partial_z^k \xi(\boldsymbol{t}, z)\big|_{z=\lambda}$

where $k \in \mathbb{N}$ and λ belongs to the support of C (these are polynomials when $\boldsymbol{t} \in \Gamma$)[9]. In particular if we freeze the times t_k for $k \geq 2$ we see that they depend rationally on x. Returning to $\mathrm{Gr}^{\mathrm{ad}}$ via (1.86) we get an additional factor which has the effect of making every a_j a proper rational function (see [42, Lemma 6.1]).

In the same manner, when C is homogeneous the i-th column of the matrix M defined in (1.88) is made up of polynomials multiplied by $g(\lambda_i) = \mathrm{e}^{\xi(\boldsymbol{t},\lambda_i)}$; these exponentials can be factored out to find that τ is the determinant of a matrix of polynomials in the $\partial_z^k \xi(\boldsymbol{t}, z)\big|_{z=\lambda}$ multiplied by the exponential of a linear function of x:

$$\tau_U = p(\boldsymbol{t}) \prod_{i=1}^{d} \mathrm{e}^{\xi(\boldsymbol{t},\lambda_i)} \tag{1.90}$$

(this already makes $u = 2\partial_x^2 \log \tau$ into a rational function). To get back from $U = (C, z^d)^* \in \mathrm{Gr}^{\mathrm{rat}}$ to $W = (C, q_C)^* \in \mathrm{Gr}^{\mathrm{ad}}$ we must act with

$$\eta = \frac{z^d}{\prod_{i=1}^{d}(z-\lambda_i)} = \prod_{i=1}^{d} \frac{z}{z-\lambda_i} = \prod_{i=1}^{d} \frac{1}{1-\lambda_i z^{-1}} = \prod_{i=1}^{d} \frac{1}{q_z(\lambda_i)} \tag{1.91}$$

(notice that here the λ_i are not necessarily distinct); but recall from (1.79) that $q_z(\lambda_i)^{-1} = \exp \sum_{k\geq 0} (\lambda_i^k/k) z^{-k}$, hence

$$\hat{\eta} = \prod_{i=1}^{d} \mathrm{e}^{-\sum_{k\geq 0} \lambda_i^k t_k} = \prod_{i=1}^{d} \mathrm{e}^{-\xi(\boldsymbol{t},\lambda_i)}$$

This factor exactly cancels the corresponding one in (1.90), hence

$$\tau_W = p(\boldsymbol{t})$$

which is a polynomial of degree d in x and degree less than or equal to d for any other time.

To establish theorem 1.85 it would remain to prove one of the converse implications, e.g. that only the points of $\mathrm{Gr}^{\mathrm{ad}}$ give rise to proper rational solutions; this needs a more technical (and rather unilluminating) argument for which we refer the reader to [42].

1.4 The scalar KP/CM correspondence

In this section we define, following [43], a map

$$\beta \colon \mathcal{C} \to \mathrm{Gr}^{\mathrm{ad}}$$

that will turn out to be bijective and equivariant with respect to the CM flows (on $\mathcal{C}$) and the KP flows (on $\mathrm{Gr}^{\mathrm{ad}}$). This gives a purely geometric interpretation of the correspondence between the motion of the poles of rational KP solutions and the motion of CM particles.

[9] More precisely we have $\partial_z^k g = g(z) P_k(\xi')$, where P_k is the k-th Faà di Bruno differential polynomial (see e.g. [12]) evaluated on $\xi'(\boldsymbol{t}, z) = x + 2t_2 z + 3t_3 z^2 + \ldots$.

1.4.1 Simple points

A point $\mathcal{W} \in \mathrm{Gr}^{\mathrm{Ad}}$ with support Λ will be called **simple** if, for every $\lambda \in \Lambda$, the subspace W_λ belongs to the (only) 1-dimensional cell of Gr_λ. Recalling the description of 1-point Grassmannians in subsection 1.2.2, this means that

$$W_\lambda = \mathrm{span}\{\frac{1}{z-\lambda} + \alpha\} \oplus (z-\lambda)\mathcal{P} \tag{1.92}$$

hence each W_λ is completely determined by a single complex number α. Now if $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ then the abstract Grassmannian coordinates of $\mathcal{W}$ are simply $(\alpha_1, \ldots, \alpha_n)$, where α_i is the above-defined parameter for the subspace W_{λ_i}.

We remark that, from the point of view of the dual mapping, the subspace (1.92) is the annihilator of a condition of the form $c = \mathrm{ev}_{1,\lambda} - \alpha\, \mathrm{ev}_{0,\lambda}$, i.e. the simplest possible kind of 1-point condition. By an easy calculation we see that the action of the adjoint involution on a subspace of the form (1.92) only involves a change of sign for α:

$$W_\lambda^* = \mathrm{span}\{\frac{1}{z-\lambda} - \alpha\} \oplus (z-\lambda)\mathcal{P} \tag{1.93}$$

so that the embedding (1.46) maps $\mathcal{W} \in \mathrm{Gr}^{\mathrm{Ad}}$ to the subspace $W \in \mathrm{Gr}^{\mathrm{ad}}$ consisting of rational functions which are regular except for (at most) simple poles at $\{\lambda_1, \ldots, \lambda_n\}$ (and a pole of any order at infinity) and satisfying the n conditions (1.48):

$$\operatorname*{res}_{z=\lambda_i} \left(\frac{1}{z-\lambda_i} - \alpha_i\right) f(z)\mathrm{d}z = 0 \quad \text{for all } i = 1 \ldots n$$

We can calculate the (reduced) Baker function corresponding to such spaces, and the result is as follows [43, Sect. 3]: let's define $Y := \mathrm{diag}(-\lambda_1, \ldots, -\lambda_n)$ and let X be the Moser matrix associated to $(\lambda_1, \ldots, \lambda_n)$ with diagonal entries $(-\alpha_1, \ldots, -\alpha_n)$; finally set $w = v^\top = \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}$. Then

$$\tilde{\psi}_W(\boldsymbol{t}, z) = 1 - w(zI + Y)^{-1}(\boldsymbol{t}.X)^{-1}v \tag{1.94}$$

where $\boldsymbol{t}.X$ denotes exactly the action (1.20) on X, i.e.

$$\boldsymbol{t}.X = X + \sum_{i\geq 1} it_i(-Y)^{i-1} = X + x - 2t_2Y + \ldots$$

Formula (1.94) effectively defines a correspondence between the space $\mathcal{C}_n''$ for every $n \in \mathbb{N}$ and the subset of $\mathrm{Gr}^{\mathrm{ad}}$ consisting of simple points with support of cardinality n (it is easy to check that the expression on the right-hand side of (1.94) is invariant under the $\mathrm{GL}(n)$-action (1.6), so that it really determines a point in $\mathcal{C}_n''$). Moreover this correspondence is bijective because both spaces are coordinatized by the set $\{\lambda_1, \ldots, \lambda_n\}$ and the n-tuple $(\alpha_1, \ldots, \alpha_n)$.

From (1.94) (or by a similar calculation) also follows an explicit expression for the tau function associated to W in terms of the corresponding Calogero-Moser matrices:

$$\tau_W(\boldsymbol{t}) = \det \boldsymbol{t}.X \tag{1.95}$$

This immediately imply that the correspondence is equivariant, since the action of the generic CM flow $\boldsymbol{t}$ on X is exactly the same as the action of the KP flow $\boldsymbol{t}$ on the tau function τ_W (cfr. (1.82)).

1.4.2 Extension to $\mathcal{C}$

It remains to extend the correspondence defined above to a map β defined on the whole of $\mathcal{C}_n$. The strategy followed in [43] to achieve this goal is a bit tricky, but the final result is very neat:

Theorem 1.96 (Prop. 4.7 in [43])**.** *The map defined on each* $\mathcal{C}''_n$ *by* (1.94) *extends to a map* $\beta\colon \mathcal{C} \to \mathrm{Gr}^{\mathrm{ad}}$ *which is continuous and* Γ*-equivariant.*

It remains to show that this extended β is still a bijection. In order to do this let's define the following partition of $\mathrm{Gr}^{\mathrm{ad}}$:

$$\mathrm{Gr}^{\mathrm{ad}}(n) := \{\, W \in \mathrm{Gr}^{\mathrm{ad}} \mid \tau_W \text{ is a polynomial of degree } n \text{ in } x \,\}$$

Then by (1.95) it is clear that β maps $\mathcal{C}_n$ into $\mathrm{Gr}^{\mathrm{ad}}(n)$.

Theorem 1.97. *For each* $n \in \mathbb{N}$ *the restriction of* β *to* $\mathcal{C}_n$ *is a bijection.*

To see that this is the case consider first the open subset $\mathcal{C}'_n$ where X is diagonalizable, and denote correspondingly by $\mathrm{Gr}^{\mathrm{ad}}(n)'$ the subset of $\mathrm{Gr}^{\mathrm{ad}}(n)$ consisting of those subspaces W such that the polynomial τ_W has n distinct roots in x; then by (1.95) we have that β maps $\mathcal{C}'_n$ into $\mathrm{Gr}^{\mathrm{ad}}(n)'$. By definition, for every $W \in \mathrm{Gr}^{\mathrm{ad}}(n)'$ we can write its tau function in the form[10]

$$\tau_W(\boldsymbol{t}) = \prod_{i=1}^{n}(x + x_i(\boldsymbol{t}')) \tag{1.98}$$

in such a manner that the functions $x_i(\boldsymbol{t}')$ are distinct for $\boldsymbol{t}' = \mathbf{0}$, hence also for all sufficiently small $\boldsymbol{t}'$. We map this subspace to the point $(X, Y, v, w) \in \mathcal{C}'_n$ as per theorem 1.16 where $X = \mathrm{diag}(x_1(\mathbf{0}), \ldots, x_n(\mathbf{0}))$ and Y is the associated Moser matrix with diagonal entries

$$Y_{ii} = -\frac{1}{2}\frac{\partial x_i}{\partial t_2}(\mathbf{0})$$

We claim that the map γ so defined is an inverse for β. Indeed:

- Starting from the point $(X, Y, v, w) \in \mathcal{C}'_n$ described by the $2n$ parameters (x_i, p_i) we arrive via β to the tau function (1.95), that we write (by freezing the times from t_3 on)

$$\tau_W(x, t_2) = \det((X - 2t_2 Y) + xI)$$

[10] We denote by $\boldsymbol{t}'$ the times t_k with $k > 1$.

The roots (in x) of this polynomial, let's call them $\lambda_i(t_2)$, are the opposite of the eigenvalues of the matrix $X - 2t_2Y$. Now applying γ we get the conditions

$$\begin{cases} \lambda_i(0) = x_i \\ -\frac{1}{2}\partial_2\lambda_i(0) = p_i \end{cases}$$

The first of them is trivial, and the second is true by definition of the momenta for Calogero-Moser particles.

- Starting from a tau function of the form (1.98), let $\bar{\tau}$ be the tau function of $\beta(\gamma(W))$ and $\bar{x}_i$ the corresponding roots; we have to show that $x_i = \bar{x}_i$ identically in $\boldsymbol{t}'$. As before, we certainly have $x_i(\mathbf{0}) = \bar{x}_i(\mathbf{0})$ and similarly $\partial_2 x_i(\mathbf{0}) = \partial_2\bar{x}_i(\mathbf{0})$; but the t_2-evolution of x_i is given by the Calogero-Moser equation, and by the Γ-equivariance of β the same holds for $\bar{x}_i$. This enables us to deduce that $\tau(x, t_2, \mathbf{0}) = \bar{\tau}(x, t_2, \mathbf{0})$ (using the uniqueness theorem for ODEs and analytic continuation), and a lemma by Shiota [40] guarantees that the tau function of a point in $\mathrm{Gr}^{\mathrm{ad}}$ is completely determined by its dependence on the first two times.

It remains to prove the bijectivity of β on the singular locus $\mathcal{C}_n \setminus \mathcal{C}'_n$; this is done using the properties (established in full generality by theorem 1.96) of continuity and equivariance of β. For the details please refer to [43].

Chapter 2

The multicomponent KP/CM correspondence

In this Chapter we start by showing (Section 2.1) how a multicomponent version of the rational Calogero-Moser system may be obtained by a process of Hamiltonian reduction which is a direct (albeit not straightforward) generalization of the one employed in Section 1.1. Then in Section 2.2 we define the multicomponent analogue of the infinite-dimensional Grassmannians of Section 1.2 and in Section 2.3 we explain their relationships with solutions of the multicomponent KP hierarchy. Finally in Section 2.4 we focus on the reduction to the matrix KP hierarchy: we prove the rationality of the solutions coming from the multicomponent adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}(m)$ and reinterpret the multicomponent KP/CM correspondence as a bijection between (two suitable subspaces of) the phase space of multicomponent CM system and $\mathrm{Gr}^{\mathrm{ad}}(m)$.

2.1 Multicomponent Calogero-Moser spaces

In [13] Gibbons and Hermsen introduced a multicomponent generalization of the rational CM system. The idea is to increase the number of degrees of freedom of the system by associating to each particle a complex m-dimensional vector for some $m \geq 1$; as we will see later, the case $m = 1$ corresponds to the standard CM system.

Let's denote by e_i the (column) vector associated to the i-th particle and by f_i the corresponding (row) vector of momenta. We identify the product of n copies of $\mathbb{C}^m$ with the space of $m \times n$ matrices and write the configuration space of the new system as

$$Q_{n,m} := \mathbb{C}^{(n)}_{\mathrm{reg}} \times \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$$

On the corresponding phase space

$$T^*Q_{n,m} = \mathbb{C}^{(n)}_{\mathrm{reg}} \times \mathbb{C}^n \times \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) \times \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$$

we have the canonical symplectic structure

$$\omega = \sum_{i=1}^{n} (\mathrm{d}p_i \wedge \mathrm{d}q_i + \mathrm{d}f_i \wedge \mathrm{d}e_i) \tag{2.1}$$

As Hamiltonian we take

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\langle f_i, e_j\rangle\langle f_j, e_i\rangle}{(q_i - q_j)^2} \tag{2.2}$$

The resulting equations of motion are

$$\ddot{q}_k = -2\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{\langle f_k, e_i\rangle\langle f_i, e_k\rangle}{(q_k - q_i)^3} \tag{2.3}$$

$$\dot{e}_k = -\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{\langle f_i, e_k\rangle}{(q_k - q_i)^2} e_i \tag{2.4}$$

$$\dot{f}_k = \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{\langle f_k, e_i\rangle}{(q_k - q_i)^2} f_i \tag{2.5}$$

Notice that the evolution of the e_i and f_j remains in the linear subspace of $\mathbb{C}^m$ spanned by their initial values, thus we can require without loss of generality $m \leq n$.

Consider now the n^2 quantities $f_{ij} := \langle f_i, e_j\rangle$. Their evolution is given by

$$\dot{f}_{ij} = \langle \dot{f}_i, e_j\rangle + \langle f_i, \dot{e}_j\rangle = \sum_{k\neq i} \frac{f_{ik} f_{kj}}{q_{ki}^2} - \sum_{k\neq j} \frac{f_{kj} f_{ik}}{q_{kj}^2} \tag{2.6}$$

In particular we have $\dot{f}_{ii} = 0$, so the n quantities $\langle f_i, e_i\rangle$ for $i = 1 \dots n$ are constants of motion. It is then natural to restrict the system to the $2nm$-dimensional invariant submanifold of $T^*Q_{n,m}$ defined by the equations $\langle f_i, e_i\rangle = c$ for some constant c independent of i; in the following we will take

$$\langle f_i, e_i\rangle = 1 \tag{2.7}$$

Then equations (2.6) can be rewritten

$$\dot{f}_{ij} = \sum_{k\neq i,j} f_{ik} f_{kj} \left(\frac{1}{q_{ki}^2} - \frac{1}{q_{kj}^2} \right) \tag{2.8}$$

We will call the dynamical system so obtained the **rational m-component Calogero-Moser system**. In [13] it is proven that this model retains all the good properties of its scalar counterpart, i.e. it is completely integrable and also explicitly solvable via purely algebraic methods.

We now generalize the quotient construction of Section 1.1 to the multicomponent CM system, again following [13] (see also [43, Sect. 8]).

Take two natural numbers $n, m \geq 1$ with $m \leq n$ and define

$$U_{n,m} := \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) \tag{2.9}$$

Elements of this space are pairs (X, W) made by a $n \times n$ matrix and a $m \times n$ matrix. The cotangent bundle $T^*U_{n,m}$ is then isomorphic to

$$V_{n,m} := \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) \oplus \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \tag{2.10}$$

This is a complex vector space of dimension $2n^2 + 2nm$; a point $p \in V_{n,m}$ is a quadruple (X, Y, V, W) comprising two $n \times n$ matrices, a $n \times m$ matrix and a $m \times n$ matrix, respectively.

On $V_{n,m}$ we take the symplectic form

$$\omega = \mathrm{tr}(\mathrm{d}Y \wedge \mathrm{d}X + \mathrm{d}V \wedge \mathrm{d}W) \tag{2.11}$$

Consider now, as in the scalar case, the action of the group $\mathrm{GL}(n, \mathbb{C})$ on $U_{n,m}$ given by

$$G.(X, V) = (GXG^{-1}, WG^{-1}) \tag{2.12}$$

We can lift this action on $V_{n,m}$ obtaining

$$G.(X, Y, V, W) = (GXG^{-1}, GYG^{-1}, GV, WG^{-1}) \tag{2.13}$$

and the associated momentum map is

$$J(X, Y, V, W) = [X, Y] - VW \tag{2.14}$$

We consider the inverse image of the point $-I \in \mathfrak{gl}(n)$ and define

$$\tilde{\mathcal{C}}_{n,m} := \{\, (X, Y, V, W) \mid [X, Y] - VW = -I \,\} \tag{2.15}$$

The stabilizer of $-I$ is the whole group $\mathrm{GL}(n, \mathbb{C})$, so the Marsden-Weinstein procedure gives the reduced space

$$\mathcal{C}_{n,m} := \tilde{\mathcal{C}}_{n,m}/\mathrm{GL}(n, \mathbb{C}) \tag{2.16}$$

that we will call the **n-particle, m-components Calogero-Moser space**.

A general study of the manifolds (2.16) seems to be significantly more difficult compared to the case $m = 1$. To show why this is so let us perform an analysis of the momentum map equation

$$[X, Y] - VW = -I \tag{2.17}$$

analogous to the one that, for standard Calogero-Moser spaces, enables us to prove theorems 1.14 and 1.16. Take a point $p = (\tilde{X}, \tilde{Y}, \tilde{V}, \tilde{W}) \in \tilde{\mathcal{C}}_{n,m}$ such that one of the first two matrices, $\tilde{X}$ say, is diagonalizable, and let $G \in \mathrm{GL}(n)$ be a matrix such that $G\tilde{X}G^{-1} = \mathrm{diag}(x_1, \ldots, x_n)$. We consider the quadruple $(X, Y, V, W) := G.(\tilde{X}, \tilde{Y}, \tilde{V}, \tilde{W})$ and split the corresponding identity (2.17) in its diagonal and off-diagonal parts:

$$\sum_{k=1}^{m} V_{ik} W_{ki} = 1 \tag{2.18}$$

$$(x_i - x_j)Y_{ij} = \sum_{k=1}^{m} V_{ik}W_{kj} \tag{2.19}$$

Now when $m = 1$ the n equations (2.18) tell us that no entry of V or W (hence of VW) can be zero, and comparing with the equations (2.19) this implies that $x_i - x_j \neq 0$ for every $i \neq j$, i.e. that the eigenvalues of X are pairwise distinct (this is indeed part of the statement of theorem 1.16). On the other hand no such conclusion can be drawn when $m > 1$, since it is entirely possible that both members of (2.19) separately vanish.

To avoid this problem we put ourselves in the dense open subset

$$\tilde{\mathcal{C}}'_{n,m} := \{\, (X, Y, V, W) \in \tilde{\mathcal{C}}_{n,m} \mid X \text{ is diagonalizable with distinct eigenvalues} \,\}$$

This is clearly preserved by the action (2.13), so there is no problem to define the corresponding quotient

$$\mathcal{C}'_{n,m} := \tilde{\mathcal{C}}'_{n,m} / \operatorname{GL}(n, \mathbb{C})$$

We also define the n row vectors $f_1, \ldots, f_n$ as the rows of the matrix V (i.e., $(f_i)_j := V_{ij}$) and the n column vectors $e_1, \ldots, e_n$ as the columns of the matrix W (i.e., $(e_i)_j := W_{ji}$); then we can rewrite equations (2.18) and (2.19) as

$$\langle f_i, e_i \rangle = 1 \tag{2.20}$$

$$Y_{ij} = \frac{\langle f_i, e_j \rangle}{x_i - x_j} \tag{2.21}$$

Equations (2.20) exactly reproduce the constraint (2.7) that we imposed in defining the multicomponent CM system, whereas equations (2.21) imply that the matrix Y has the form

$$Y = \begin{pmatrix} p_1 & \frac{\langle f_1, e_2 \rangle}{x_1 - x_2} & \cdots & \frac{\langle f_1, e_n \rangle}{x_1 - x_n} \\ \frac{\langle f_2, e_1 \rangle}{x_2 - x_1} & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\langle f_n, e_1 \rangle}{x_n - x_1} & \cdots & \cdots & p_n \end{pmatrix} \tag{2.22}$$

for some complex numbers $p_1, \ldots, p_n$.

At this point we would like to spend the residual freedom in the $\operatorname{GL}(n)$-action on the quadruple (X, Y, V, W) to normalize the entries of the vectors e_i and f_i in a suitable way. Remember that the eigenvalues of X are assumed pairwise distinct, so the residual symmetry after diagonalization of X is generated by the action of a diagonal matrix D and a permutation matrix P; the latter only controls the ordering of the particles. With regard to D, first of all we remark that the constraints (2.20) imply that no f_i and no e_i can be the zero vector; this means that every e_i, say, has at least a nonzero entry that we can normalize to 1 using the action of D, and then we can use the n constraints (2.20) to normalize also the corresponding entry in f_i. Notice that when $m = 1$ this completely fixes every entry of V and W to 1 (as per the statement of theorem 1.16), so these two matrices effectively disappear from the picture.

As in the scalar case, we now express the restriction to $\mathcal{C}'_{n,m}$ of the reduced symplectic form on $\mathcal{C}_{n,m}$ using the coordinate system consisting of the $2n$ complex numbers x_i, p_i and the $2n(m-1)$ unconstrained entries of e_i and f_i; we get

$$\omega(x,p,e,f) = \sum_{i=1}^{n} (\mathrm{d}p_i \wedge \mathrm{d}x_i + \mathrm{d}e_i \wedge \mathrm{d}f_i) \tag{2.23}$$

which is exactly the symplectic form (2.1).

We consider for each $k \geq 1$ the flows on $V_{n,m}$ generated by the Hamiltonians $H_k = \mathrm{tr}(-Y)^k$; again the equations of motion are trivially integrated to give

$$t_k.(X,Y,V,W) = (X + kt_k(-Y)^{k-1}, Y, V, W) \tag{2.24}$$

These flows are GL(n)-invariant and we can project them down to $\mathcal{C}_{n,m}$ and, by restriction, to $\mathcal{C}'_{n,m}$; in particular for $k=2$ we have

$$H_2 = \sum_{i=1}^{n} p_i^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\langle f_i, e_j \rangle \langle f_j, e_i \rangle}{(q_i - q_j)^2} \tag{2.25}$$

which is twice the Hamiltonian (2.2).

2.2 Multicomponent Grassmannians

In this Section we define the multicomponent versions of the various infinite-dimensional Grassmannians considered in Section 1.2.

2.2.1 The multicomponent rational Grassmannian

We consider the space $\mathcal{R}^m$ of m-tuples of rational functions on $\mathbb{C}P^1$ with the direct sum decomposition

$$\mathcal{R}^m = \mathcal{P}^m \oplus \mathcal{R}^m_- \tag{2.26}$$

with associated canonical projection maps $\pi_+ \colon \mathcal{R}^m \to \mathcal{P}^m$ and $\pi_- \colon \mathcal{R}^m \to \mathcal{R}^m_-$. The **full m-component rational Grassmannian** $\overline{\mathrm{Gr}}^{\mathrm{rat}}(m)$ is the set of closed linear subspaces $W \subseteq \mathcal{R}^m$ for which there exist polynomials $p, q \in \mathcal{P}$ such that

$$p\mathcal{P}^m \subseteq W \subseteq q^{-1}\mathcal{P}^m \tag{2.27}$$

This is the straightforward generalization of condition (1.23). We now proceed to compute the virtual dimension of such a subspace, i.e. the index of $p_+ := \pi_+|_W$. Let's write

$$W = W' \oplus p\mathcal{P}^m \tag{2.28}$$

and set $p'_+ := p_+|_{W'}$; then by the same reasoning as in the scalar case we have

$$\ker p_+ \cong \ker p'_+ \quad \text{and} \quad \mathrm{coker}\, p_+ \cong \frac{U}{\mathrm{im}\, p'_+} \tag{2.29}$$

where U is the subspace defined by the decomposition $\mathcal{P}^m = U \oplus p\mathcal{P}^m$; notice that $\dim U = m \deg p$, so the analogue of equation (1.27) is

$$\dim \operatorname{coker} p_+ = m \deg p - \dim \operatorname{im} p'_+ \tag{2.30}$$

whence the following expression for the virtual dimension of W:

$$\operatorname{vdim} W = \dim W' - m \deg p \tag{2.31}$$

The only difference with respect to the scalar formula (1.28) is the factor m; as we will see, this factor will matter a lot.

The m-**component rational Grassmannian** $\mathrm{Gr}^{\mathrm{rat}}(m)$ is the subset of $\overline{\mathrm{Gr}}^{\mathrm{rat}}(m)$ consisting of subspaces of virtual dimension zero.

Theorem 2.32. *A subspace $W \in \overline{\mathrm{Gr}}^{\mathrm{rat}}(m)$ has virtual dimension zero if and only if the codimension of the inclusion $W \subseteq q^{-1}\mathcal{P}^m$ coincides with $m \deg q$.*

The proof goes along the same lines of the one of theorem 1.29 to get the formula

$$\operatorname{codim}_{q^{-1}\mathcal{P}^m} W = m \deg q - \operatorname{vdim} W \tag{2.33}$$

from which follows the claim.

We proceed to introduce the dual description for $\overline{\mathrm{Gr}}^{\mathrm{rat}}(m)$. The algebraic dual of $\mathcal{P}^m$ is $(\mathbb{C}^\omega)^m$, with the pairing defined in the obvious manner. For every $k \in \{1, \dots, m\}$, $r \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we have the functional $\mathrm{ev}_{k,r,\lambda}$ defined by

$$\langle \mathrm{ev}_{k,r,\lambda}, (p_1, \dots, p_m) \rangle = p_k^{(r)}(\lambda)$$

The set $\mathcal{E}^{(m)}$ of such functionals is linearly independent; we denote by $\mathscr{C}^{(m)}$ the linear space they generate and take it as our new space of differential conditions. We also put

$$\mathscr{C}_\lambda^{(m)} := \operatorname{span}\{\mathrm{ev}_{k,r,\lambda}\}_{1 \le k \le m,\, r \in \mathbb{N}}$$

and

$$\mathscr{C}_{r,\lambda}^{(m)} := \operatorname{span}\{\mathrm{ev}_{k,s,\lambda}\}_{1 \le k \le m,\, 0 \le s < r}$$

with the usual convention $\mathscr{C}_{0,\lambda}^{(m)} = \{0\}$.

Given $c \in \mathscr{C}^{(m)}$ we continue to call the finite set of points $\lambda \in \mathbb{C}$ such that the projection of c on $\mathscr{C}_\lambda^{(m)}$ is nonzero the *support* of c. The annihilators of the subspaces $C \subseteq \mathscr{C}^{(m)}$ and $V \subseteq \mathcal{P}^m$ are defined just as in the scalar case.

Theorem 2.34. *A subspace $W \subseteq \mathcal{R}^m$ belongs to $\overline{\mathrm{Gr}}^{\mathrm{rat}}(m)$ if and only if there exists a finite-dimensional subspace $C \subseteq \mathscr{C}^{(m)}$ and a polynomial q such that $W = q^{-1}V_C$; moreover $W \in \mathrm{Gr}^{\mathrm{rat}}(m)$ if and only if $m \deg q = \dim C$.*

The proof is exactly analogous to the one of theorem 1.32; the last statement follows from (2.33), which reads

$$\operatorname{vdim} W = m \deg q - \dim C \tag{2.35}$$

We remark that whereas in the scalar case ($m = 1$), given the space of conditions C, we can always choose the polynomial q in such a manner that the resulting subspace W has virtual dimension zero, this is no longer possible when $m > 1$, since $\operatorname{vdim} W = 0$ implies $\dim C = m \deg q$ and this is only possible when m divides the dimension of C. We conclude that, when $m > 1$, not every set of differential conditions on $\mathcal{P}^m$ determines a subspace of virtual dimension zero.

We define the dual multicomponent rational Grassmannian as

$$\mathrm{Gr}^{\mathrm{rat}*}(m) := \left\{ (C, q) \in \mathrm{Gr}_{\mathrm{fin}} \mathscr{C}^{(m)} \times \mathcal{P} \;\middle|\; \begin{array}{l} \dim C = dm \text{ for some } d \in \mathbb{N} \\ q \text{ monic of degree } d \end{array} \right\} \tag{2.36}$$

Then theorem 2.34 gives a surjective mapping $\mathrm{Gr}^{\mathrm{rat}*}(m) \to \mathrm{Gr}^{\mathrm{rat}}(m)$ defined by $(C, q)^* = q^{-1} V_C$ which is the m-component version of the dual map of subsection 1.2.1.

To derive the analogue of theorem 1.36 we introduce the group $\Gamma_-(m)$ as the direct product of m copies of Γ_-, seen as a group of diagonal $m \times m$ matrices. This group acts on $\mathrm{Gr}^{\mathrm{rat}}(m)$ by matrix multiplication from the right: given $h \in \Gamma_-(m)$ with $h = \operatorname{diag}(h_1, \ldots, h_m)$, $h_\alpha \in \Gamma_-$ we have

$$(f_1, \ldots, f_m).h = (h_1 f_1, \ldots, h_m f_m)$$

We can see that this action is free by applying component-wise the same argument that works in the scalar case.

Theorem 2.37. *The images by the dual map of two points with the same conditions space C lie in the same $\Gamma_-(m)$-orbit of $\mathrm{Gr}^{\mathrm{rat}}(m)$.*

Indeed given the two subspaces $W_1 = (C, q_1)^*$ and $W_2 = (C, q_2)^*$ in $\mathrm{Gr}^{\mathrm{rat}}(m)$ the matrix $\eta := \frac{q_1}{q_2} I_m$ belongs to $\Gamma_-(m)$ (notice that q_1 and q_2 are both of degree $m \dim C$) and is such that $W_1 \eta = W_2$.

2.2.2 The multicomponent adelic Grassmannian

We now turn to the problem of defining the multicomponent analogues of 1-point Grassmannians and of the abstract adelic Grassmannian.

Given $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$ we denote by $\mathrm{Gr}_{\lambda,k}(m)$ the set of linear subspaces $W \in \mathrm{Gr}^{\mathrm{rat}}(m)$ for which we can choose $p = q = (z - \lambda)^k$:

$$(z - \lambda)^k \mathcal{P}^m \subseteq W \subseteq (z - \lambda)^{-k} \mathcal{P}^m \tag{2.38}$$

By theorem 2.32 the codimension of W in $(z - \lambda)^{-k} \mathcal{P}^m$ is mk and by equation (2.31) the codimension of $(z - \lambda)^k \mathcal{P}^m$ in W is also mk. We conclude that each $W \in \mathrm{Gr}_{\lambda,k}(m)$ is uniquely determined by the mk-dimensional linear subspace W' in the $2mk$-dimensional space

$$\frac{(z-\lambda)^{-k}\mathcal{P}^m}{(z-\lambda)^k\mathcal{P}^m} \tag{2.39}$$

In other words, $\mathrm{Gr}_{\lambda,k}(m) \cong \mathrm{Gr}(mk, 2mk)$.

We now specialize as usual to the $\lambda = 0$ case; everything we say will also hold for any other choice of λ. We consider the family of finite-dimensional Grassmannians $\{\mathrm{Gr}_{0,k}\}_{k\geq 0}$ and define a map $f_k\colon\ \mathrm{Gr}_{0,k}(m) \to \mathrm{Gr}_{0,k+1}(m)$ for any $k \in \mathbb{N}$ as follows: given $W \in \mathrm{Gr}_{0,k}(m)$ of the form

$$W = \mathrm{span}\{w_1, \ldots, w_{mk}\} \oplus z^k\mathcal{P}^m$$

we rewrite it as

$$W = \mathrm{span}\{w_1, \ldots, w_{mk}, e_1 z^k, \ldots, e_m z^k\} \oplus z^{k+1}\mathcal{P}^m$$

and see it as a subspace in $z^{-k-1}\mathcal{P}^m \subseteq \mathcal{R}^m$; quotienting by $z^{k+1}\mathcal{P}^m$ we obtain the corresponding point in $\mathrm{Gr}_{0,k+1}$. This gives us an inductive system of sets $\{\mathrm{Gr}_{0,k}(m)\}_{k\geq 0}$ and maps $\{f_k\}_{k\geq 0}$; we define the m-**component 1-point Grassmannian** at 0 as the corresponding direct limit:

$$\mathrm{Gr}_0(m) := \varinjlim_{k\in\mathbb{N}} \mathrm{Gr}_{0,k}(m) \tag{2.40}$$

We claim that $\mathrm{Gr}_0(m)$ is isomorphic as a set to Gr_0, i.e. that there exists a bijection between them. Indeed this follows immediately from the fact that the two inductive systems $\{\mathrm{Gr}(k, 2k)\}_{k\geq 0}$ and $\{\mathrm{Gr}(mk, 2mk)\}_{k\geq 0}$ are cofinal (see [7, Ch. III, §7, Prop. 8]). This result is not completely unexpected in view of the "scalarization" isomorphism that will be introduced in the next subsection.

We now analyze the cell structure of $\mathrm{Gr}_0(m)$. Recall that in subsection 1.2.2 we used the functions $\{z^k\}_{k\in\mathbb{Z}}$ to induce a canonical basis in every linear space $z^{-k}\mathcal{P}/z^k\mathcal{P}$. Similarly in this setting we can use the vector-valued functions $\{e_i z^k\}_{1\leq i\leq m,\, k\in\mathbb{Z}}$ (where e_i is the m-component vector with 1 in the i-th entry and zero elsewhere) to induce the basis

$$\begin{matrix} (e_1 z^{k-1}, & & e_1 z, & e_1, & e_1 z^{-1}, & & e_1 z^{-k}, \\ \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots \\ e_m z^{k-1}, & & e_m z, & e_m, & e_m z^{-1}, & & e_m z^{-k}) \end{matrix}$$

in $z^{-k}\mathcal{P}^m/z^k\mathcal{P}^m$. We again consider the associated complete flag defined by

$$V_{mj+i} := \mathrm{span}\{e_r z^s\}_{1\leq r\leq i,\, k-1\geq s\geq k-1-j}$$

for all $1 \leq i \leq m$ and $0 \leq j \leq 2k-1$, with the corresponding decomposition of $\mathrm{Gr}(mk, 2mk)$ in Schubert cells. A point in $\mathrm{Gr}_0(m)$ is still described by a partition p and a point $\vec{\alpha} \in \mathbb{C}^{|p|}$; to recover the linear subspace of $\mathcal{R}^m$ associated to this data, let k be the least natural number such that the Young diagram of p is contained in a square of side mk and select the corresponding Schubert cell C_σ in $\mathrm{Gr}(mk, 2mk)$; then the point

$\vec{\alpha} \in C_\sigma$ determines a subspace $W' \subseteq z^{-k}\mathcal{P}^m/z^k\mathcal{P}^m$, and the subspace we are looking for is $W = W' \oplus z^k\mathcal{P}^m$.

At this point we can define the m-component abstract adelic Grassmannian by recycling definition 1.40, i.e. as the restricted product

$$\mathrm{Gr}^{\mathrm{Ad}}(m) := \prod_{\lambda\in\mathbb{C}}{}' \mathrm{Gr}_\lambda(m) \tag{2.41}$$

with origin $\mathcal{P}^m$. Notice that again, as a set, this is exactly the same as the "scalar" adelic Grassmannian $\mathrm{Gr}^{\mathrm{Ad}}$; the only difference lies in the interpretation of its points. For example, in subsection 1.4.1 we saw that the subspaces in Gr_λ associated to the (only) weight 1 partition (1) are of the form

$$W = \mathrm{span}\{\frac{1}{z-\lambda} + \alpha\} \oplus (z-\lambda)\mathcal{P}$$

for some $\alpha \in \mathbb{C}$; they are the annihilators of $c = \mathrm{ev}_{1,\lambda} - \alpha\,\mathrm{ev}_{0,\lambda}$. On the contrary in $\mathrm{Gr}_\lambda(2)$ the point determined by the partition (1) is

$$W = \mathrm{span}\{(1,0), (\frac{1}{z-\lambda}, \alpha)\} \oplus (z-\lambda)\mathcal{P}^2 \tag{2.42}$$

which corresponds to the annihilator of the subspace generated by the two conditions $c_1 = \mathrm{ev}_{2,0,\lambda}$ and $c_2 = \mathrm{ev}_{2,1,\lambda} - \alpha\,\mathrm{ev}_{1,0,\lambda}$.

We proceed to generalize the embedding $\mathrm{Gr}^{\mathrm{Ad}} \to \mathrm{Gr}^{\mathrm{rat}}$ described in subsection 1.2.3. For every $\lambda \in \mathbb{C}P^1$ and for any pair of vectors of rational functions $f, g \in \mathcal{R}^m$ we define the symmetric bilinear form

$$\langle f, g\rangle_\lambda := \operatorname*{res}_{z=\lambda}(f \cdot g)(z)\mathrm{d}z$$

and denote as usual by $\mathrm{Ann}_\lambda W$ the annihilator of a subspace $W \subseteq \mathcal{R}^m$; moreover we define $W^* := \mathrm{Ann}_\infty W$. By the same calculations done in the scalar case we see that $\mathcal{P}^m$ and $\mathcal{R}^m_-$ are two maximal isotropic subspaces for $\langle\,\cdot\,,\,\cdot\,\rangle_\infty$, and that the correspondence $W \mapsto W^*$ defines an involution on $\mathrm{Gr}^{\mathrm{rat}}(m)$ that preserves each $\mathrm{Gr}_\lambda(m)$.

We can now define an embedding $i\colon \mathrm{Gr}^{\mathrm{Ad}}(m) \to \mathrm{Gr}^{\mathrm{rat}}(m)$ that generalizes the one defined by equation (1.46): given $\mathcal{W} = \{W_\lambda\}_{\lambda\in\mathbb{C}} \in \mathrm{Gr}^{\mathrm{Ad}}(m)$ we put

$$\mathcal{W} \mapsto W := \bigcap_{\lambda\in\mathbb{C}} \mathrm{Ann}_\lambda W^*_\lambda \tag{2.43}$$

Again, if the support of $\mathcal{W}$ consists of a single point then by (1.47) we just reproduce the trivial embedding $\mathrm{Gr}_\lambda(m) \to \mathrm{Gr}^{\mathrm{rat}}(m)$. In the general case, for each $\lambda \in \Lambda$ take a natural number k_λ such that $W^*_\lambda \in \mathrm{Gr}_{\lambda,k_\lambda}(m)$; then there exists a basis for W^*_λ of the form

$$\{\,\omega_{11}, \dots, \omega_{1m}, \dots, \omega_{k_\lambda,1}, \dots, \omega_{k_\lambda,m}, e_1(z-\lambda)^{k_\lambda}, \dots, e_m(z-\lambda)^{k_\lambda}, \dots\}$$

where each ω_{ij} is a m-component vector of Laurent polynomials in $z-\lambda$. Then $\mathrm{Ann}_\lambda W_\lambda^*$ consists of all $f=(f_1,\dots,f_m)\in\mathcal{R}^m$ such that each component function f_i has a pole of order at most k_λ at λ and such that the following mk_λ conditions are satisfied:

$$\langle f,\omega_{ij}\rangle_\lambda = 0 \qquad \text{for all } 1\le j\le m,\ 1\le i\le k_\lambda \tag{2.44}$$

Each of these conditions amounts to a homogeneous linear condition on the coefficients of the Laurent series of the f_i's at λ, i.e. to an element of $\mathscr{C}_\lambda^{(m)}$.

Summing up, $i(\mathcal{W})$ is the space of m-tuples of rational functions which are regular except for a pole at ∞ and a pole of order at most k_λ at λ, and satisfying all the conditions (2.44) at the various points λ in the support of $\mathcal{W}$.

We denote by $\mathrm{Gr}^{\mathrm{ad}}(m)$ the image of i in $\mathrm{Gr}^{\mathrm{rat}}(m)$. In terms of the dual map we have the following equivalent description: let C be a (finite-dimensional) homogeneous space of conditions in $\mathscr{C}^{(m)}$ and define a polynomial q_C as in (1.49) but taking $n_\lambda := (\dim C_\lambda)/m$, then

$$\mathrm{Gr}^{\mathrm{ad}}(m) = \{\, W\in\mathrm{Gr}^{\mathrm{rat}}(m) \mid W=(C,q)^* \text{ with } C \text{ homogeneous and } q=q_C \,\}$$

2.2.3 The multicomponent Segal-Wilson Grassmannian

We define the **m-component Segal-Wilson Grassmannian** to be the Grassmannian defined as in subsection 1.2.4 but using the Hilbert space

$$H^{(m)} = L^2(S^1,\mathbb{C}^m)$$

with the obvious splitting $H^{(m)} = H_+^{(m)}\oplus H_-^{(m)}$ in the two subspaces consisting of functions with only positive (resp. negative) Fourier coefficients. More precisely, we again view the elements of $H^{(m)}$ as functions $\mathbb{C}\to\mathbb{C}^m$ by embedding S^1 in $\mathbb{C}$ as the circle γ_R, and denote $H^{(m)}(R)$ the corresponding Hilbert space; then we denote by $\mathrm{Gr}(m,R)$ the (index zero component of the) Segal-Wilson Grassmannian of $H^{(m)}(R)$.

The previous definition is well known in the literature, and indeed it is explicitly used already in [39], where it serves an essential rôle in defining the solution spaces for the various reductions of the KP hierarchy (notably the KdV hierarchy). However, Segal and Wilson consistently employ a trick that we call *scalarization*: namely, they consider the map $H^{(m)}\to H^{(1)}$ defined by sending the basis $\{e_i z^k\}_{i=1\dots m,\,k\in\mathbb{Z}}$ for $H^{(m)}$ to the basis $\{z^k\}_{k\in\mathbb{Z}}$ for $H^{(1)}$ in the following manner:

$$e_i z^k \mapsto z^{mk+i-1} \tag{2.45}$$

In other words, to a vector-valued function $f\colon\mathbb{C}\to\mathbb{C}^m$ with $f=(f_1,\dots,f_m)$ they assign the scalar-valued function $\hat f$ defined by "interleaving" its Fourier coefficients:

$$\hat f(z) = f_1(z^m) + z f_2(z^m) + \cdots + z^{m-1} f_m(z^m)$$

This map is bijective: indeed given $\hat f\in H^{(1)}$ we can recover the components of f as

$$f_{i+1}(z) = \frac{1}{m}\sum_{\{\zeta\mid\zeta^m=z\}} \zeta^{-i}\hat f(\zeta)$$

Furthermore, the map (2.45) is an isometry and preserves every plausible property of functions (continuity, differentiability, polinomiality, etc.). Anyway we prefer not to adopt this point of view, since it clearly destroys the geometry of the problem under consideration. On the other hand, the mere existence of such an isomorphism clarifies why the multicomponent versions of the 1-point Grassmannians and of the adelic Grassmannian do not contain any more points relative to their scalar counterparts: indeed the two are automatically in one-to-one correspondence via the map (2.45).

The relationship between the m-component Segal-Wilson Grassmannian and the m-component rational Grassmannian is of course exactly analogous to the one holding in the scalar case: given $W \in \mathrm{Gr}^{\mathrm{rat}}(m)$ we choose $R \in \mathbb{R}^+$ such that every root of the polynomial q appearing in (2.27) is contained in the open disc $|z| < R$; then the restrictions $f|_{\gamma_R}$ for all $f \in W$ determine a linear subspace whose L^2-closure belongs to $\mathrm{Gr}(m, R)$. Again, this embedding automatically defines a topology on $\mathrm{Gr}^{\mathrm{rat}}(m)$ and its subspaces by restriction.

We next define the group $\Gamma_+(m, R)$ as the direct product of m copies of $\Gamma_+(R)$, embedded in the loop group $\mathrm{LGL}(m, \mathbb{C})$ (which naturally acts on $\mathrm{Gr}(m)$) as the subgroup of diagonal matrices. Explicitly, we can write $g \in \Gamma_+(m, R)$ as

$$g = \mathrm{diag}(g_1, \ldots, g_m) \quad \text{with } g_\alpha \in \Gamma_+(R)$$

If we take the elements of $\mathrm{Gr}(m, R)$ to be row vectors for definiteness, we thus have an action of $\Gamma_+(m, R)$ on $\mathrm{Gr}(m, R)$ by matrix multiplication from the right:

$$\begin{pmatrix} f_1 & \ldots & f_m \end{pmatrix} \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_m \end{pmatrix} = \begin{pmatrix} g_1 f_1 & \ldots & g_m f_m \end{pmatrix}$$

Using the representation (1.51) for the m functions g_i we see that every element $g \in \Gamma_+(m, R)$ is described by a set of coefficients $\{\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(m)}\}$ via the equality

$$g = \mathrm{diag}(1 + \sum_{k \geq 1} h_k^{(1)} z^k, \ldots, 1 + \sum_{k \geq 1} h_k^{(m)} z^k) \tag{2.46}$$

Or we can use the representation (1.52), in which case we will write

$$g = \exp \mathrm{diag}(\sum_{k \geq 1} t_k^{(1)} z^k, \ldots, \sum_{k \geq 1} t_k^{(m)} z^k) \tag{2.47}$$

Then g is described by the family of coefficients $\{\boldsymbol{t}^{(1)}, \ldots, \boldsymbol{t}^{(m)}\}$.

2.3 Multicomponent KP hierarchy

In this Section we briefly introduce the multicomponent KP hierarchy, referring the reader to [11, 18] for a detailed analysis.

2.3.1 Definition

Given $m \geq 1$ we denote by $\mathcal{A}_m$ the differential algebra of $m \times m$ matrices with entries belonging to an algebra of smooth functions of a variable x (with $D = \frac{\partial}{\partial x}$) and m further families of variables

$$\boldsymbol{t}^{(1)} := \{t_i^{(1)}\}_{i \geq 1} \quad \dots \quad \boldsymbol{t}^{(m)} := \{t_i^{(m)}\}_{i \geq 1}$$

In the sequel we will often use the shorthand notation $\bar{\boldsymbol{t}} := \{\boldsymbol{t}^{(1)}, \dots, \boldsymbol{t}^{(m)}\}$. Now let $\Psi(\mathcal{A}_m)$ be the ring of pseudo-differential operators with coefficients in $\mathcal{A}_m$, and denote by $G^{(m)}$ the group of pseudo-differential operator of the form

$$\phi = I_m + \sum_{k \geq 1} W_k D^{-k} \tag{2.48}$$

We define the corresponding “dressed” operator as

$$Q := \phi D \phi^{-1} = D + \sum_{i \geq 1} U_i D^{-i} \tag{2.49}$$

Moreover, for every $\alpha \in \{1, \dots, m\}$ we define

$$R_\alpha := \phi E_\alpha \phi^{-1} \tag{2.50}$$

where E_α is the matrix with 1 in the entry (α, α) and zero elsewhere. Note that

$$[Q, R_\alpha] = 0 \qquad \text{for all } \alpha \in \{1, \dots, m\}$$

since $[Q, R_\alpha] = \phi(DE_\alpha - E_\alpha D)\phi^{-1} = 0$; moreover we have

$$[R_\alpha, R_\beta] = 0 \tag{2.51a}$$

$$\sum_{\alpha=1}^{m} R_\alpha = \phi I_m \phi^{-1} = I_m \tag{2.51b}$$

We can now define the **multicomponent KP flow** on $G^{(m)}$ with respect to the variable $t_k^{(\alpha)}$ by the following equation:

$$\partial_{k\alpha} \phi = -(\phi D^k \phi^{-1} R_\alpha)_- \phi \tag{2.52}$$

where $\partial_{k\alpha} := \frac{\partial}{\partial t_k^{(\alpha)}}$, as usual, is understood to act on the coefficients of a pseudo-differential operator. This gives an infinite family of (matrix) partial differential equations in the unknowns $\{W_i\}_{i \geq 1}$; notice that (2.52) reduces to (1.57) when $m = 1$ (taking $R_1 = 1$).

The evolution equation (2.52) determines the following evolution equations for the operators Q and R_α:

$$\partial_{k\alpha} Q = [(Q^k R_\alpha)_+, Q] \tag{2.53a}$$

$$\partial_{k\alpha} R_\beta = [(Q^k R_\alpha)_+, R_\beta] \tag{2.53b}$$

This is the Lax form of the multicomponent KP hierarchy; again equation (2.53a) determines an infinite family of (matrix) partial differential equations in the unknowns $\{U_i\}_{i\geq 1}$.

Finally let's come to formal Baker functions. Given the set of variables $\bar{\boldsymbol{t}}$ we define the matrix-valued map

$$\Xi(\bar{\boldsymbol{t}}, z) := \operatorname{diag}(\xi(\boldsymbol{t}^{(1)}, z), \ldots, \xi(\boldsymbol{t}^{(m)}, z)) = \begin{pmatrix} \sum_{i\geq 0} t_i^{(1)} z^i & & \\ & \ddots & \\ & & \sum_{i\geq 0} t_i^{(m)} z^i \end{pmatrix} \tag{2.54}$$

and consider the free rank 1 module over $\Psi(\mathcal{A}_m)$ consisting of formal expressions of the form

$$\psi = \tilde{\psi} e^{\Xi(\bar{\boldsymbol{t}}, z)}$$

where $\tilde{\psi} \in \mathcal{A}_m((z^{-1}))$ is a formal Laurent series in z^{-1} with coefficients in $\mathcal{A}_m$. We define an action of $\Psi(\mathcal{A}_m)$ on these object as follows:

$$D.(\tilde{\psi} e^{\Xi(\bar{\boldsymbol{t}}, z)}) := (\tilde{\psi}' + z\tilde{\psi}) e^{\Xi(\bar{\boldsymbol{t}}, z)}$$

whence

$$D^n.(\tilde{\psi} e^{\Xi(\bar{\boldsymbol{t}}, z)}) = \sum_{k\geq 0} \binom{n}{k} \tilde{\psi}^{(k)} z^{n-k} \, e^{\Xi(\bar{\boldsymbol{t}}, z)} \tag{2.55}$$

for all $n \in \mathbb{Z}$.

Now given $\phi \in G^{(m)}$ the expression

$$\psi(\bar{\boldsymbol{t}}, z) := \phi . e^{\Xi(\bar{\boldsymbol{t}}, z)} = (I_m + \sum_{i\geq 1} W_i z^{-i}) e^{\Xi(\bar{\boldsymbol{t}}, z)} \tag{2.56}$$

will be called the **formal Baker function** associated to ϕ. It can be shown that for every $\phi \in G^{(m)}$ the associated formal Baker function ψ is an eigenfunction of the operator Q with respect to the eigenvalue z,

$$Q\psi = z\psi \tag{2.57}$$

Moreover if ϕ is a solution of (2.52) then

$$\partial_{k\alpha}\psi = (Q^k R_\alpha)_+ \psi \tag{2.58}$$

so that the multicomponent KP hierarchy may be characterized as the compatibility condition for the linear system of equations (2.57) and (2.58).

At this point we would like to remark that it is not trivial to explicitly write down even the first instances of the system of partial differential equations (2.52) (or (2.53)): some of the simplest cases, along with a number of useful reductions of this hierarchy, are studied in [18].

2.3.2 Solutions of multicomponent KP and $\mathrm{Gr}(m)$

The relationship between the multicomponent Segal-Wilson Grassmannian of subsection 2.2.3 and the multicomponent KP hierarchy introduced above is very similar to the one holding between their scalar counterparts, explained in subsection 1.3.3.

Let's start with some conventions. As in subsection 1.3.3, we let $\mathrm{Gr}(m)$ and $\Gamma_+(m)$ stand respectively for $\mathrm{Gr}(m, R)$ and $\Gamma_+(m, R)$ for some fixed choice of R. Moreover, given a matrix-valued function $\psi(z)$ we will say that ψ belongs to a subspace $W \in \mathrm{Gr}(m)$ if and only if each row of ψ, seen as an element of $H^{(m)}$, belongs to W.

For every $W \in \mathrm{Gr}(m)$ we define

$$\Gamma_+(m)^W := \{\, g \in \Gamma_+(m) \mid Wg^{-1} \text{ is transverse} \,\} \tag{2.59}$$

where of course "transverse" here means that the orthogonal projection $Wg^{-1} \to H_+^{(m)}$ is an isomorphism.

Now in $H_+^{(m)}$ we have the m elements $\{e_1, \ldots, e_m\}$ (i.e., the canonical basis of $\mathbb{C}^m$ multiplied by the identity function); we define the **reduced Baker function** associated to W and g to be the matrix-valued function ψ whose row ψ_α is the inverse image of $e_\alpha \in H_+^{(m)}$ by $\pi_+|_{Wg^{-1}}$. By definition it has the form

$$\tilde{\psi}_W = I_m + \sum_{i \geq 1} W_i(g) z^{-i} \tag{2.60}$$

for some matrices $\{W_i\}_{i \geq 1}$ whose entries are functions of g. Now, since each row of the matrix $\tilde{\psi}_W$ belongs to the subspace Wg^{-1}, each row of the product matrix $\tilde{\psi}_W g$ will belong to the subspace W we started with; the **Baker function** associated to W is the map ψ_W which sends $g \in \Gamma_+(m)^W$ to this matrix:

$$\psi_W(g, z) = \left(I_m + \sum_{i \geq 1} W_i(g) z^{-i} \right) g(z) \tag{2.61}$$

As in the scalar case, if we express g in terms of the family of coefficients $\bar{\boldsymbol{t}}$ as defined by equation (2.47) then the expression (2.61) defines an element of the group $G^{(m)}$,

$$\phi_W := I_m + \sum_{i \geq 1} W_i(\bar{\boldsymbol{t}}) D^{-i} \tag{2.62}$$

such that $\phi_W . \mathrm{e}^{\Xi(\bar{\boldsymbol{t}}, z)} = \psi_W$.

Theorem 2.63. *The operator defined by* (2.62) *satisfies the multicomponent KP equation* (2.52).

For the proof see [11]. We thus conclude that to each point $W \in \mathrm{Gr}(m)$ we can associate a solution to the multicomponent KP hierarchy, equivalently expressed by the Baker function ψ_W or by the pseudo-differential operator ϕ_W.

To recover the corresponding order one operator Q_W that solves equation (2.53) we must use the matrix version of the dressing relations (1.65):

$$U_1 = -W_1' \tag{2.64a}$$
$$U_2 = -W_2' - U_1 W_1 \tag{2.64b}$$
$$U_3 = -W_3' - U_2 W_1 + U_1 W_1' - U_1 W_2 \tag{2.64c}$$

and so on (notice that in the present setting the ordering of factors matters). Again, the correspondence is not one-to-one: elements of $G^{(m)}$ which are related by "gauge transformations" of the form $\phi \mapsto \phi(I_m + \sum_{i\geq 1} C_i z^{-i})$, where the C_i's are constant elements of $\mathcal{A}_m$, correspond to the same Q.

2.3.3 The tau function

The tau function for the multicomponent KP hierarchy has been introduced by Dickey in [11]. Again, we do not need the precise details of its definition; the important fact is that for each subspace $W \in \mathrm{Gr}(m, R)$ we can define

- a holomorphic function τ_W on $\Gamma_+(m, R)$, and
- for each pair of indices $\alpha, \beta \in \{1, \ldots, m\}$ with $\alpha \neq \beta$, a holomorphic function $\tau_{W\alpha\beta}$ also on $\Gamma_+(m, R)$,

all determined up to constant factors. We then have the following m-component analogue of Sato's formula (1.78):

$$\tilde{\psi}_W(g, z)_{\alpha\beta} = \begin{cases} \dfrac{\tau_W(g q_{z\alpha})}{\tau_W(g)} & \text{if } \alpha = \beta \\ z^{-1}\dfrac{\tau_{W\alpha\beta}(g q_{z\beta})}{\tau_W(g)} & \text{if } \alpha \neq \beta \end{cases} \tag{2.65}$$

where for each $z \in \mathbb{C}$ and $\alpha \in \{1, \ldots, m\}$ we define $q_{z\alpha}$ to be the element of $\Gamma_+(m, R)$ that has q_z at the (α, α) entry and 1 elsewhere on the diagonal.

If we write g in terms of the coefficients $\bar{\boldsymbol{t}}$ defined by equation (2.47) then (2.65) becomes

$$\tilde{\psi}_W(\bar{\boldsymbol{t}}, z)_{\alpha\beta} = \begin{cases} \dfrac{\tau_W(\bar{\boldsymbol{t}} - [z^{-1}]_\alpha)}{\tau_W(\bar{\boldsymbol{t}})} & \text{if } \alpha = \beta \\ z^{-1}\dfrac{\tau_{W\alpha\beta}(\bar{\boldsymbol{t}} - [z^{-1}]_\beta)}{\tau_W(\bar{\boldsymbol{t}})} & \text{if } \alpha \neq \beta \end{cases} \tag{2.66}$$

where we have defined the multicomponent Miwa shift as

$$\bar{\boldsymbol{t}} - [z^{-1}]_\alpha := \{t_k^{(\gamma)} - \delta_{\alpha\gamma}\frac{1}{kz^k}\}_{k\geq 1,\, \gamma = 1\ldots m}$$

Expanding the numerators in (2.66) in a Taylor series around $z = \infty$ we can get explicit formulas for the (matrix) coefficients of the Baker function associated to W (hence for the corresponding solution to equation (2.52)); for example we have

$$W_{1\alpha\beta}(\bar{\boldsymbol{t}}) = \begin{cases} -\dfrac{1}{\tau_W}\dfrac{\partial \tau_W}{\partial t_1^{(\alpha)}} & \text{if } \alpha = \beta \\ \dfrac{\tau_{W\alpha\beta}}{\tau_W} & \text{if } \alpha \neq \beta \end{cases} \tag{2.67}$$

$$W_{2\alpha\beta}(\bar{\boldsymbol{t}}) = \begin{cases} \dfrac{1}{2\tau_W}\left(\dfrac{\partial^2 \tau_W}{(\partial t_1^{(\alpha)})^2} - \dfrac{\partial \tau_W}{\partial t_2^{(\alpha)}}\right) & \text{if } \alpha = \beta \\ -\dfrac{1}{\tau_W}\dfrac{\partial \tau_{W\alpha\beta}}{\partial t_1^{(\beta)}} & \text{if } \alpha \neq \beta \end{cases} \tag{2.68}$$

and so on. These expressions may be used, via (2.49) and (2.50), to obtain also the operators Q and R_α (solutions of (2.53)) directly in terms of the tau functions.

2.3.4 Some rational solutions

In this subsection we look for solutions to the m-component KP hierarchy that depend rationally on the times $t_1^{(1)}, \dots, t_1^{(m)}$. Following the trail of the scalar case, we are led to investigate the conditions under which the entries of the (matrix) coefficients of the operator Q_W (or equivalently ϕ_W) associated to a point $W \in \mathrm{Gr}^{\mathrm{rat}}(m)$ are proper rational functions of the variables $t_1^{(\alpha)}$. Anyway we have to be a little careful here, since *a priori* it is not at all clear if such a request makes sense or not.

To clarify the matter consider the Lax equation (2.53a) for the rather trivial case $k = 0$ (we are temporarily supposing that the elements of our differential algebra $\mathcal{A}_m$ depend on the m additional times $t_0^{(\alpha)}$); since $(Q^0 R_\gamma)_+ = E_\gamma$, it reads

$$\frac{\partial U_k}{\partial t_0^{(\gamma)}} = [E_\gamma, U_k] \tag{2.69}$$

that is,

$$\partial_{0\gamma} u_{k\alpha\beta} = \delta_{\alpha\gamma} u_{k\gamma\beta} - \delta_{\gamma\beta} u_{k\alpha\gamma}$$

for any $\alpha, \beta \in \{1, \dots, m\}$, $k \geq 1$. Thus the diagonal elements of U_k are constant with respect to the times $t_0^{(\gamma)}$ (this corresponds to the fact that the time variable t_0 plays no rôle in the scalar setting), whereas for the generic off-diagonal element $u_{\alpha\beta}^k$ with $\alpha \neq \beta$ we have an exponential dependence of the form

$$u_{k\alpha\beta} = \mathrm{e}^{t_0^{(\alpha)} - t_0^{(\beta)}}$$

This fact suggests that it is unreasonable to require every entry of U_k to depend rationally on the times $t_1^{(\gamma)}$; we should rather expect a rational dependence for the diagonal elements, and a dependence of the form $\frac{g_\alpha}{g_\beta} f(\bar{\boldsymbol{t}})$ (with f rational) for the entry at position (α, β) with $\alpha \neq \beta$.

With this idea in mind we turn to the multicomponent versions of formulas (1.87) and (1.88). Consider first the problem of determining the Baker function of a point in $\mathrm{Gr}^{\mathrm{rat}}(m)$: as in the scalar case, theorem 2.37 coupled with the transformation law

$$\psi_{W\eta}(g,z) = \psi_W(g\eta^{-1},z)\cdot\eta \qquad \text{for every } \eta\in\Gamma_-(m) \tag{2.70}$$

implies that it is enough to consider the case $q = z^d$. So take a space of conditions $C \subseteq \mathscr{C}^{(m)}$ of dimension md and consider the corresponding subspace $W = \left(C, z^d\right)^* \in \mathrm{Gr}^{\mathrm{rat}}(m)$; we claim that ψ_W must have the form

$$\psi_W(g,z) = \left(I_m + \sum_{j=1}^{d} W_j(g)z^{-j}\right)g(z) \tag{2.71}$$

Indeed by the usual arguments (cfr. [42, Sect. 4]) we have on the one hand that each row of the matrix-valued function $z^d\psi_W(g,z)$ (for every fixed g) belongs to the L^2-closure of V_C in $H_+^{(m)}(R)$ for some R, and on the other hand that each functional $\mathrm{ev}_{k,r,\lambda}$ extends uniquely to a continuous functional on $H_+^{(m)}(R)$; hence it must be $(z^d\psi_W)_\alpha \in V_C$ for every $\alpha\in\{1,\dots,m\}$. Now,

$$(z^d\psi_W)_{\alpha\beta} = \left(z^d\delta_{\alpha\beta} + \sum_{j\geq 1} W_{j\alpha\beta}(g)z^{d-j}\right)g_\beta(z)$$

By expressing g_β using the $\boldsymbol{h}$ coefficients via (2.46) and imposing that every matrix element so obtained is a polynomial we see that every W_j with $j > d$ must be the zero matrix, hence the expression (2.71).

To determine the matrices $\{W_1,\dots,W_d\}$ let $(c_1,\dots,c_{md})$ be a basis for C; for each $\alpha\in\{1,\dots,m\}$ we take the α-th row of $z^d\psi_W$ and impose the equalities $\langle c_i,(z^d\psi_W)_\alpha\rangle = 0$ (with $i\in\{1,\dots,md\}$). This gives the following linear system of equations:

$$\langle c_i, \left(z^d\delta_{\alpha\beta} + \sum_{j=1}^{d} W_{j\alpha\beta}(g)z^{d-j}\right)g_\beta(z)\rangle = 0 \tag{2.72}$$

In other words, we have a family of m linear systems, each of which involves md equations, for a total of m^2d scalar equations. The unknowns are of course the m^2 entries of the d matrices $\{W_1,\dots,W_d\}$; the coefficients of these unknowns involve, as in the scalar case, the g_β's and their derivatives evaluated at the points in the support of C.

The calculation of the tau function associated to $\left(C,z^d\right)^* \in \mathrm{Gr}^{\mathrm{rat}}(m)$ is done in [11] (indeed it serves there as a motivating example for the general theory), and the result is the following. The "diagonal" tau function τ_W is given by the determinant

$$\tau_W(g) = \det(\langle c_i, z^{j-1}g_\gamma\rangle)_{\substack{i=1\dots md\\ j=1\dots d,\,\gamma=1\dots m}} \tag{2.73}$$

(notice that here $z^{j-1}g_\gamma$ must really be interpreted as the row vector having $z^{j-1}g_\gamma$ in its γ-th entry and zero elsewhere). More explicitly, the tau function is the determinant of the following block matrix:

$$\tau_W(g) = \begin{vmatrix} \langle c_1, g_1\rangle & \dots & \langle c_{md}, g_1\rangle \\ \vdots & & \vdots \\ \langle c_1, g_m\rangle & \dots & \langle c_{md}, g_m\rangle \\ \langle c_1, zg_1\rangle & \dots & \langle c_{md}, zg_1\rangle \\ \vdots & & \vdots \\ \langle c_1, zg_m\rangle & \dots & \langle c_{md}, zg_m\rangle \\ \vdots & \vdots & \vdots \\ \langle c_1, z^{d-1}g_1\rangle & \dots & \langle c_{md}, z^{d-1}g_1\rangle \\ \vdots & & \vdots \\ \langle c_1, z^{d-1}g_m\rangle & \dots & \langle c_{md}, z^{d-1}g_m\rangle \end{vmatrix} \tag{2.74}$$

As in the scalar case, this is just the matrix of coefficients of the linear system (2.72), so that the system is determined exactly when $g \in \Gamma_+(m)^W$. Observe that an expression such as $z^{j-1}g_\gamma$ may equivalently be read as $\partial_{1\gamma}^{j-1}g_\gamma$.

The "off-diagonal" tau functions $\tau_{W\alpha\beta}$ are given as follows: consider the family of $(md+1)\times(md+1)$ matrices indexed by $\alpha, \beta \in \{1, \dots, m\}$

$$M_{\alpha\beta} := \begin{pmatrix} \langle c_1, g_1\rangle & \dots & \langle c_{md}, g_1\rangle & 0 \\ \vdots & & \vdots & z^{-d} \\ \langle c_1, g_m\rangle & \dots & \langle c_{md}, g_m\rangle & 0 \\ \langle c_1, zg_1\rangle & \dots & \langle c_{md}, zg_1\rangle & 0 \\ \vdots & & \vdots & z^{1-d} \\ \langle c_1, zg_m\rangle & \dots & \langle c_{md}, zg_m\rangle & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \langle c_1, z^{d-1}g_1\rangle & \dots & \langle c_{md}, z^{d-1}g_1\rangle & 0 \\ \vdots & & \vdots & z^{-1} \\ \langle c_1, z^{d-1}g_m\rangle & \dots & \langle c_{md}, z^{d-1}g_m\rangle & 0 \\ \langle c_1, z^{d}g_\alpha\rangle & \dots & \langle c_{md}, z^{d}g_\alpha\rangle & \delta_{\alpha\beta} \end{pmatrix} \tag{2.75}$$

where in the last column the only nonzero element is on the β-th row of each block. Then $\tau_{W\alpha\beta}$ is the cofactor of the element z^{-1} in the last column. Explicitly:

$$\tau_{W\alpha\beta}(g) = (-1)^{m-\beta}\det(\langle c_i, z^{j-1}(g_\gamma + \delta_{jd}\delta_{\gamma\beta}(zg_\alpha - g_\gamma))\rangle)_{\substack{i=1\dots md \\ j=1\dots d,\, \gamma=1\dots m}} \tag{2.76}$$

We can now study the rationality of Baker functions coming from the points of the m-component adelic Grassmannian defined in subsection 2.2.2. Let's consider first the case in which W belongs to a 1-point Grassmannian.

Theorem 2.77. *Let $W \in \mathrm{Gr}_\lambda(m)$, then:*

1) *Each diagonal entry of the reduced Baker function* $\tilde{\psi}_W$ *is a rational function of the times* $t_1^{(1)}, \dots, t_1^{(m)}$ *that tends to* 1 *as* $t_1^{(\alpha)} \to \infty$ *for all* α*;*
2) *The tau function* τ_W *is a polynomial in* $t_1^{(1)}, \dots, t_1^{(m)}$ *with constant leading coefficient.*

Proof. Just as in the scalar case, the two statements are equivalent by virtue of (the diagonal part of) Sato's formula (2.65), so it suffices to prove one of them; we choose the first. By hypothesis we have $W = \big(C, (z-\lambda)^d\big)^*$ with $C \subseteq \mathscr{C}_\lambda^{(m)}$ of dimension md; let $(c_1, \dots, c_{md})$ be a basis for it. Define the subspace $U := \big(C, z^d\big)^* \in \mathrm{Gr}^{\mathrm{rat}}(m)$; its tau function is given by (2.73). To compute the diagonal elements of the corresponding Baker function we use Sato's formula for U:

$$\tilde{\psi}_{U\alpha\alpha} = \frac{\tau_U(g q_{\zeta\alpha})}{\tau_U(g)} \tag{2.78}$$

(here we use ζ as a parameter and z as a variable). Since we are only interested in the times with subscript 1 we will work in the stationary setting, i.e. we put $t_k^{(\alpha)} = 0$ for every $k \geq 2$, $\alpha = 1 \dots m$.

Now, each c_i is a condition supported at λ, hence we can define a family of polynomials $\{\phi_{i\gamma}\}_{i=1\dots md,\, \gamma=1\dots m}$ (with $\phi_{i\gamma}$ only depending on $t_1^{(\gamma)}$) by the equality

$$\langle c_i, g_\gamma \rangle = g_\gamma(\lambda)\phi_{i\gamma} \tag{2.79}$$

To apply (2.78) we need to know the determinant of the matrices $\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma\rangle$ and $\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma(1-\delta_{\alpha\gamma}\frac{z}{\zeta})\rangle$. As regards the first, using (2.79) its generic element is written

$$\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma\rangle = \partial_{1\gamma}^{j-1}(g_\gamma(\lambda)\phi_{i\gamma}) = g_\gamma(\lambda)(\partial_{1\gamma}+\lambda)^{j-1}\phi_{i\gamma} \tag{2.80}$$

For the second matrix we have

$$\partial_{1\gamma}^{j-1}\langle c_i, g_\gamma(1-\delta_{\alpha\gamma}\frac{z}{\zeta})\rangle = \partial_{1\gamma}^{j-1}\langle c_i, g_\gamma\rangle - \partial_{1\gamma}^{j-1}\langle c_i, \delta_{\alpha\gamma} g_\gamma \frac{z}{\zeta}\rangle \tag{2.81}$$

The first term is exactly (2.80), whereas the second is

$$\delta_{\alpha\gamma}\partial_{1\gamma}^{j}\langle c_i, g_\gamma\rangle\zeta^{-1} = \delta_{\alpha\gamma}\partial_{1\gamma}^{j}(g_\gamma(\lambda)\phi_{i\gamma})\zeta^{-1} = \delta_{\alpha\gamma} g_\gamma(\lambda)(\partial_{1\gamma}+\lambda)^j\phi_{i\gamma}\zeta^{-1}$$

putting all together, equation (2.81) becomes

$$g_\gamma(\lambda)(\partial_{1\gamma}+\lambda)^{j-1}\big(\phi_{i\gamma} - \delta_{\alpha\gamma}(\partial_{1\gamma}\phi_{i\gamma} + \lambda\phi_{i\gamma})\zeta^{-1}\big)$$

that we can rewrite as

$$g_\gamma(\lambda)(1-\delta_{\alpha\gamma}\frac{\lambda}{\zeta})(\partial_{1\gamma}+\lambda)^{j-1}\big(\phi_{i\gamma} - \delta_{\alpha\gamma}\frac{1}{\zeta-\lambda}\partial_{1\gamma}\phi_{i\gamma}\big) \tag{2.82}$$

But $(1-\delta_{\alpha\gamma}\frac{\lambda}{\zeta}) = q_{\zeta\alpha}(\lambda)_\gamma$, so plugging (2.80) and (2.82) into (2.78) we obtain

$$\tilde{\psi}_{U\alpha\alpha}(g,\zeta) = (q_\zeta(\lambda))^d \frac{\det\Big((\partial_{1\gamma}+\lambda)^{j-1}(\phi_{i\gamma} - \delta_{\alpha\gamma}\frac{1}{\zeta-\lambda}\partial_{1\gamma}\phi_{i\gamma})\Big)}{\det\big((\partial_{1\gamma}+\lambda)^{j-1}\phi_{i\gamma}\big)} \tag{2.83}$$

Now, the factor $(q_\zeta(\lambda))^d$ disappears when we go back from U to W and we are left with the ratio of two determinants of matrices with polynomial entries in the times $t_1^{(\gamma)}$; this clearly gives a rational function. Moreover, if we expand the numerator of (2.83) by linearity over the sum we see that the term obtained by always choosing $\phi_{i\gamma}$ exactly reproduces the polynomial at the denominator, and all the other terms involve a polynomial which has degree strictly lower than τ in some $t_1^{(\gamma)}$ (since we substitute $\phi_{i\gamma}$ with one of its derivatives); this proves that $\tilde{\psi}_{W\alpha\alpha} \to 1$ as all the $t_1^{(\alpha)}$ tend to infinity. □

For the elements of $\mathrm{Gr}^{\mathrm{ad}}(m)$ whose support involves more than a single point the previous proof breaks down. To see why this is so consider for example a point $W \in \mathrm{Gr}^{\mathrm{ad}}(m)$ obtained by imposing a single set of m conditions $\{c_{1\lambda_i}, \dots, c_{m\lambda_i}\}$ in each point of its support $\{\lambda_1, \dots, \lambda_n\}$ (the general case is not essentially more complicated). Let's define for each α and each λ_i the $m \times m$ matrices

$$\langle C_{\lambda_i}, g_\alpha\rangle = \begin{pmatrix} \langle c_{1\lambda_i}, g_\alpha\rangle & \dots & \langle c_{m\lambda_i}, g_\alpha\rangle \\ \vdots & \ddots & \vdots \\ \langle c_{1\lambda_i}, z^{m-1} g_\alpha\rangle & \dots & \langle c_{m\lambda_i}, z^{m-1} g_\alpha\rangle \end{pmatrix} \tag{2.84}$$

Equivalently, these can be seen as the Wronskian matrices determined by the m functions $\langle c_{1\lambda_i}, g_\alpha\rangle, \dots, \langle c_{m\lambda_i}, g_\alpha\rangle$ with respect to the variable $t_1^{(\alpha)}$:

$$\begin{pmatrix} \langle c_{1\lambda_i}, g_\alpha\rangle & \dots & \langle c_{m\lambda_i}, g_\alpha\rangle \\ \vdots & \ddots & \vdots \\ \partial_{1\alpha}^{m-1}\langle c_{1\lambda_i}, g_\alpha\rangle & \dots & \partial_{1\alpha}^{m-1}\langle c_{m\lambda_i}, g_\alpha\rangle \end{pmatrix}$$

Now consider the subspace $U \in \mathrm{Gr}^{\mathrm{rat}}(m)$ determined by the same conditions but with $q = z^n$. Its tau function is given by (2.74) and, after a suitable sequence of row exchanges, we are left with the determinant of a matrix in the block form

$$\tau_U = \begin{vmatrix} \langle C_{\lambda_1}, g_1\rangle & \dots & \langle C_{\lambda_n}, g_1\rangle \\ \vdots & \ddots & \vdots \\ \langle C_{\lambda_1}, g_m\rangle & \dots & \langle C_{\lambda_n}, g_m\rangle \end{vmatrix} \tag{2.85}$$

Now from each block in this matrix we can factor out a term $g_\alpha(\lambda_i)$, but this clearly does not make the whole matrix into a matrix of polynomials; instead we have that

$$\tau_U = \begin{vmatrix} g_1(\lambda_1)\langle C_{\lambda_1}, g_1\rangle & g_1(\lambda_2)\langle C_{\lambda_2}, g_1\rangle & \dots & g_1(\lambda_n)\langle C_{\lambda_n}, g_1\rangle \\ g_2(\lambda_1)\langle C_{\lambda_1}, g_2\rangle & g_2(\lambda_2)\langle C_{\lambda_2}, g_2\rangle & \dots & g_2(\lambda_n)\langle C_{\lambda_n}, g_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ g_m(\lambda_1)\langle C_{\lambda_1}, g_m\rangle & g_m(\lambda_2)\langle C_{\lambda_2}, g_m\rangle & \dots & g_m(\lambda_n)\langle C_{\lambda_n}, g_m\rangle \end{vmatrix} \tag{2.86}$$

However, if we now suppose that $g_1 = g_2 = \dots = g_m$ (i.e. that $g = \tilde{g} I_m$ for some $\tilde{g} \in \Gamma_+$) then we can again collect all the exponential factors out of the matrix (proceeding by columns, not by rows as in the previous proof) and express τ_W as the determinant of a matrix with polynomials entries; then we can expect the rational solutions of the corresponding sub-hierarchy to be parametrized by the points of the multicomponent adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}(m)$. This result will be established in subsection 2.4.2.

2.3.5 Examples

We now explicitly show some rational solutions of the m-component KP hierarchy coming from points in $\mathrm{Gr}_\lambda(m)$. Let's start from the case $m=2$ and take just the simplest possible element of $\mathrm{Gr}_\lambda(2)$; as we know from subsection 2.2.2, this is given by the subspace (2.42) for some $\alpha \in \mathbb{C}$. The corresponding reduced Baker function reads

$$\tilde{\psi}(g,z) = I_2 + \begin{pmatrix} 0 & 0 \\ \frac{1}{\alpha}\frac{g_2(\lambda)}{g_1(\lambda)} & 0 \end{pmatrix} \frac{1}{z-\lambda}$$

and the tau functions are

$$\tau(g) = \alpha \qquad \tau_{12}(g) = 0 \qquad \tau_{21}(g) = \frac{g_2(\lambda)}{g_1(\lambda)}$$

This turns out to be a rather dull solution: the diagonal components are stationary, the $(1,2)$-component is zero and the $(2,1)$-component evolves only in the trivial way.

To get a more interesting solution we consider a subspace coming from the cell of maximal dimension in $\mathrm{Gr}_{\lambda,1}(2)$; this is the set of subspaces with Schubert symbol $\sigma = (34)$ that corresponds to the partition $p(\sigma) = (22)$ of weight 4. A generic member of this cell is described by a point $\alpha \in \mathbb{C}^4$ which is naturally seen as a 2×2 matrix $(\alpha_{ij})_{i,j=1\ldots 2}$, corresponding to the left half of the matrix representative of this point in $\mathrm{Gr}(2,4)$ (recall that we are using the canonical basis defined in subsection 2.2.2):

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & 1 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 1 \end{pmatrix}$$

These abstract Grassmannian coordinates determine the subspace

$$W = \mathrm{span}\{(\frac{1}{z-\lambda} + \alpha_{11}, \alpha_{12}), (\alpha_{21}, \frac{1}{z-\lambda} + \alpha_{22})\} \oplus (z-\lambda)\mathcal{P}^2$$

in $\mathrm{Gr}_\lambda(2)$, which is the annihilator of the linear subspace $C \subseteq \mathscr{C}^{(2)}$ generated by the two conditions

$$\begin{cases} c_1 = \mathrm{ev}_{11\lambda} - \alpha_{11}\,\mathrm{ev}_{10\lambda} - \alpha_{21}\,\mathrm{ev}_{20\lambda} \\ c_2 = \mathrm{ev}_{21\lambda} - \alpha_{12}\,\mathrm{ev}_{10\lambda} - \alpha_{22}\,\mathrm{ev}_{20\lambda} \end{cases}$$

Let's define

$$X_{\gamma\lambda} := \xi'(t^{(\gamma)}, \lambda) = t_1^{(\gamma)} + 2t_2^{(\gamma)}\lambda + 3t_3^{(\gamma)}\lambda^2 + \cdots \tag{2.87}$$

Then the tau functions associated to W are

$$\tau_W(\bar{\boldsymbol{t}}) = (X_{1\lambda} - \alpha_{11})(X_{2\lambda} - \alpha_{22}) - \alpha_{12}\alpha_{21}$$

$$\tau_{W12}(\bar{\boldsymbol{t}}) = -\frac{g_1(\lambda)}{g_2(\lambda)}\alpha_{12} \qquad \tau_{W21}(\bar{\boldsymbol{t}}) = -\frac{g_2(\lambda)}{g_1(\lambda)}\alpha_{21}$$

and the reduced Baker function is

$$\tilde{\psi}_W = I_2 - \frac{1}{\tau}\begin{pmatrix} X_{2\lambda} - \alpha_{22} & \alpha_{12}\frac{g_1(\lambda)}{g_2(\lambda)} \\ \alpha_{21}\frac{g_2(\lambda)}{g_1(\lambda)} & X_{1\lambda} - \alpha_{11} \end{pmatrix} \frac{1}{z-\lambda} \tag{2.88}$$

The previous example may be easily generalized to every value of m. Indeed consider a point in the cell of maximal dimension of $\mathrm{Gr}_{\lambda,1}(m)$ that comes from the partition $p = (m^m)$ whose Young diagram is a square of side m. Points in this cell are parametrized by a $m \times m$ matrix which is again the left half of the matrix representing the corresponding point of $\mathrm{Gr}(m, 2m)$:

$$\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1m} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \dots & \alpha_{mm} & 0 & \dots & 1 \end{pmatrix}$$

These abstract coordinates determine the linear subspace

$$W_{\lambda,\alpha} = \mathrm{span}\{\omega_1, \dots, \omega_m\} \oplus (z-\lambda)\mathcal{P}^m \tag{2.89}$$

where $\omega_1 = (\frac{1}{z-\lambda} + \alpha_{11}, \alpha_{12}, \dots, \alpha_{1m})$, $\omega_2 = (\alpha_{21}, \frac{1}{z-\lambda} + \alpha_{22}, \dots, \alpha_{2m})$ and so on up to $\omega_m = (\alpha_{m1}, \dots, \frac{1}{z-\lambda} + \alpha_{mm})$. This subspace, in turn, corresponds to the annihilator of the m conditions

$$\begin{cases} c_1 = \mathrm{ev}_{11\lambda} + a_{11}\,\mathrm{ev}_{10\lambda} + \dots + a_{1m}\,\mathrm{ev}_{m0\lambda} \\ \vdots \\ c_m = \mathrm{ev}_{m1\lambda} + a_{m1}\,\mathrm{ev}_{10\lambda} + \dots + a_{mm}\,\mathrm{ev}_{m0\lambda} \end{cases} \tag{2.90}$$

where the matrix $A = (a_{ij})$ is given by $-\alpha^\top$. Now,

$$\langle c_i, g_\gamma \rangle = g_\gamma(\lambda)(\delta_{i\gamma} X_{\gamma\lambda} + a_{i\gamma})$$

so if we define the matrix $X_\lambda := \mathrm{diag}(X_{1\lambda}, \dots, X_{m\lambda})$ then the tau functions are given by

$$\tau_W(g) = \det(X_\lambda - \alpha) \qquad \tau_{W\alpha\beta} = -\frac{g_\alpha}{g_\beta} C_{\beta,\alpha} \tag{2.91}$$

where $C_{\beta,\alpha}$ stands for the (β, α) cofactor of the matrix $X_\lambda - \alpha$. Consequently,

$$\tilde{\psi}_W(g, z) = I_m - \frac{1}{\tau} \begin{pmatrix} C_{11} & C_{21}\frac{g_1(\lambda)}{g_2(\lambda)} & \dots & C_{m1}\frac{g_1(\lambda)}{g_m(\lambda)} \\ C_{12}\frac{g_2(\lambda)}{g_1(\lambda)} & C_{22} & \dots & C_{m2}\frac{g_2(\lambda)}{g_m(\lambda)} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1m}\frac{g_m(\lambda)}{g_1(\lambda)} & C_{2m}\frac{g_m(\lambda)}{g_2(\lambda)} & \dots & C_{mm} \end{pmatrix} \frac{1}{z-\lambda}$$

These are the solutions coming from the "generic" points of $\mathrm{Gr}_\lambda(m)$ whose defining conditions (2.90) involve only the functionals $\mathrm{ev}_{kr\lambda}$ with $r \le 1$. Clearly there are much more complicated subspaces in $\mathrm{Gr}_\lambda(m)$, which correspond to points in higher-dimensional affine cells.

2.4 The multicomponent KP/CM correspondence

In this Section we explain how the matrix KP equation arises in the framework previously developed for the multicomponent KP hierarchy and characterize its rational solutions; we then reinterpret the computations of [13, 24] as a correspondence between $\mathcal{C}'_{n,m}$ and the space of rank 1 rational solutions of matrix KP.

2.4.1 The matrix KP hierarchy

In subsection 2.3.1 we saw that the evolution operators for the multicomponent KP hierarchy are given by

$$B_{k\alpha} := (Q^k R_\alpha)_+$$

for any $k \geq 1$. Now observe that

$$\sum_{\gamma=1}^{m} B_{k\gamma} = \sum_{\gamma=1}^{m} (Q^k R_\gamma)_+ = (Q^k \sum_{\gamma=1}^{m} R_\gamma)_+ = Q_+^k \tag{2.92}$$

by (2.51b). In particular for $k = 1$ we have $\sum_\gamma B_{1\gamma} = D$, and this means that the variable x whose D is the derivative operator can be identified with the sum of the times with subscript 1:

$$x = t_1^{(1)} + \cdots + t_1^{(m)} \tag{2.93}$$

More generally, for every $k \geq 1$ we define the new variable

$$t_k := \sum_{\gamma=1}^{m} t_k^{(\gamma)} \tag{2.94}$$

Then

$$\frac{\partial}{\partial t_k} = \sum_{\gamma=1}^{m} \partial_{k\gamma}$$

and the evolution of Q with respect to these new "times" is

$$\frac{\partial}{\partial t_k} Q = \sum_{\alpha=1}^{m} \partial_{k\alpha} Q = \sum_{\alpha=1}^{m} [B_{k\alpha}, Q] = [\sum_{\alpha=1}^{m} B_{k\alpha}, Q] = [Q_+^k, Q] \tag{2.95}$$

This is exactly the same as (1.59), so the corresponding hierarchy of equations will be identical to the classical KP hierarchy, the only difference being that the coefficients of Q are now $m \times m$ matrices (in particular they do not commute among themselves in general); this will be called the **matrix** (or noncommutative) **KP hierarchy**.

We now replicate the steps that led us to the KP equation in subsection 1.3.2. For $k = 2$ we have $Q_+^2 = D^2 + 2U_1$ and the corresponding evolution equations for U_1 and U_2 are

$$\frac{\partial U_1}{\partial t_2} = U_1'' + 2U_2' \tag{2.96}$$

$$\frac{\partial U_2}{\partial t_2} = U_2'' + 2U_3' + 2U_1U_1' + 2[U_1, U_2] \tag{2.97}$$

Similarly for $k = 3$ we have $Q_+^3 = D^3 + 3U_1D + 3(U_2 + U_1')$, whence the further equation

$$\frac{\partial U_1}{\partial t_3} = U_1''' + 3U_2'' + 3U_3' + 3U_1'U_1 + 3U_1U_1' \tag{2.98}$$

We consider the system consisting of the three equations above; again we can simplify U_3' between the second and the third, obtaining

$$\begin{cases}\partial_2 U_1 = U_1'' + 2U_2' \\ 3\partial_2 U_2 - 2\partial_3 U_1 = -2U_1''' - 3U_2'' - 6U_1'U_1 + 6[U_1, U_2]\end{cases}$$

Putting $u := 2U_1$, $w := U_2$, $y := t_2$ and $t := t_3$ we can rewrite the resulting system as follows:

$$\begin{cases}2w_x = \frac{1}{2}u_y - \frac{1}{2}u_{xx} & (2.99a)\\ 3w_y - u_t + 3w_{xx} + u_{xxx} + \frac{3}{2}u_x u - 3[u, w] = 0 & (2.99b)\end{cases}$$

We will refer to this system as the **matrix KP equation**. Alternatively, if we admit explicit integrations in our equations, we can solve (2.99a) for w

$$w = \frac{1}{4}\int^x u_y - \frac{1}{4}u_x \tag{2.100}$$

and plug this expression into (2.99b), obtaining

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u^2)_x + \frac{3}{4}\int^x u_{yy} - \frac{3}{4}[u, \int^x u_y] \tag{2.101}$$

which is also sometimes called the matrix (or noncommutative) KP equation, see e.g. [16].

Using the framework developed in the previous Sections we can build solutions to the matrix KP hierarchy starting from the points of the Segal-Wilson multicomponent Grassmannian. Indeed take $W \in \mathrm{Gr}(m)$ and consider the evolution given by an element $g \in \Gamma_+(m)$ of the form

$$g = \mathrm{diag}(\mathrm{e}^{\xi(\frac{1}{m}\boldsymbol{t},z)}, \dots, \mathrm{e}^{\xi(\frac{1}{m}\boldsymbol{t},z)}) \tag{2.102}$$

where, as always, $\boldsymbol{t} = \{t_k\}_{k\geq 1}$; with this choice the constraints (2.94) are automatically satisfied, so this action exactly describes the flows of equations (2.95) in $\mathrm{Gr}(m)$. Let's define $\tilde{g} := \mathrm{e}^{\xi(m^{-1}\boldsymbol{t},z)}$ and $h := \mathrm{e}^{\xi(\boldsymbol{t},z)}$ (so that $g = \tilde{g}I_m$ and $\tilde{g}^m = h$); then the Baker and tau functions for the matrix KP hierarchy are naturally expressed in terms of h only, since that single function completely controls the flows of the hierarchy.

2.4.2 Rationality of the solutions

We now prove that the solutions to the matrix KP hierarchy coming from points of $\mathrm{Gr}^{\mathrm{ad}}(m)$ are rational functions of the first time variable $x = t_1$.

Theorem 2.103. *Let $W \in \mathrm{Gr}^{\mathrm{ad}}(m)$, then:*
1) *$\tilde{\psi}_W(h, z)$ is a matrix-valued rational function of x that tends to I_m as $x \to \infty$:*
2) *$\tau_W(h)$ and $\tau_{W\alpha\beta}(h)$ are polynomial functions of x with constant leading coefficients.*

Proof. Take $W \in \mathrm{Gr}^{\mathrm{ad}}(m)$, then $W = (C, q_C)^*$ with $C \subseteq \mathscr{C}^{(m)}$ homogeneous. Let $(\lambda_1, \dots, \lambda_d)$ be the support of W *with each point counted according to its multiplicity* (so that the λ_i are not necessarily distinct). Then for each $i \in \{1, \dots, d\}$ we can take exactly m linearly independent conditions at λ_i that we call $(c_{ij})_{j=1\dots m}$. In this way we get a basis $(c_{11}, \dots, c_{dm})$ for C made of 1-point conditions.

Now consider the subspace $U := \left(C, z^d\right)^* \in \mathrm{Gr}^{\mathrm{rat}}(m)$; it is related to W by the following element of $\Gamma_-(m)$ (cfr. (1.91)):

$$\eta = \prod_{i=1}^{d} q_z(\lambda_i)^{-1} I_m = \exp\left(\sum_{k\geq 0}\sum_{i=1}^{d} \frac{\lambda_i^k}{k} z^{-k}\right) I_m \tag{2.104}$$

This corresponds to multiplying the tau function by

$$\hat{\eta} = \prod_{\alpha=1}^{m} \exp\left(-\sum_{k\geq 0}\sum_{i=1}^{d} \lambda_i^k t_k^{(\alpha)}\right) = \prod_{\alpha=1}^{m}\prod_{i=1}^{d} g_\alpha(\lambda_i)^{-1} = \prod_{i=1}^{d} h(\lambda_i)^{-1} \tag{2.105}$$

since $g_\alpha = \tilde{g}$ for every $\alpha \in \{1, \dots, m\}$ and $\tilde{g}^m = h$.

Let's define the family of polynomials $\{\phi_{ij\gamma}\}$ (for $i = 1 \dots d$, $j = 1 \dots m$, $\gamma = 1 \dots m$) by the equality

$$\langle c_{ij}, g_\gamma \rangle = \tilde{g}(\lambda_i)\phi_{ij\gamma} \tag{2.106}$$

Again, this works precisely because each c_{ij} is a 1-point condition; notice that, although $g_\gamma = \tilde{g}$ for every γ, the polynomials ϕ still depend on γ since it is the index of the only nonzero entry of the row vector on which c_{ij} acts.

We can now easily compute the tau functions associated to U by retracing the same steps as in the proof of theorem 2.77, but now using the polynomials defined by (2.106). Then $\tau_U(g)$ is the determinant of the matrix whose generic element is

$$\tilde{g}(\lambda_i)(\partial_{1\gamma} + \lambda_i)^{k-1}\phi_{ij\gamma}$$

The term $\tilde{g}(\lambda_i)$ does not depend on the row indices (k, γ) so we can factor it out from the determinant to obtain

$$\tau_U = \prod_{i=1}^{d} (\tilde{g}(\lambda_i))^m \det\left((\partial_{1\gamma} + \lambda_i)^{k-1}\phi_{ij\gamma}\right) \tag{2.107}$$

But $\prod_i (\tilde{g}(\lambda_i))^m = \prod_i h(\lambda_i)$ is exactly the inverse of (2.105), so that

$$\tau_W = \det\left((\partial_{1\gamma} + \lambda_i)^{k-1}\phi_{ij\gamma}\right) \tag{2.108}$$

is the determinant of a matrix with polynomial entries in $t_1^{(\alpha)} = \frac{x}{m}$, hence a polynomial in x and the coefficient of the top degree term involves only the constants λ_i. This implies by Sato's formula that $\tilde{\psi}_{W\alpha\alpha} \to 1$ as $x \to \infty$.

Now take $\alpha, \beta \in \{1, \dots, m\}$, $\alpha \neq \beta$ and consider the off-diagonal tau function $\tau_{U\alpha\beta}(g)$; it is given by the determinant of a matrix $M_{\alpha\beta}(g)$ which coincides with the one involved in the definition of $\tau_U(g)$ except for the row corresponding to $k = d - 1$, $\gamma = \beta$ which is replaced by the row $\langle c_{ij}, \partial_{1\alpha}^d g_\alpha \rangle$. But since $g_\alpha = g_\beta = \tilde{g}$ we can again collect out of the determinant the same factor as before, so that $\tau_{W\alpha\beta}(g) = \det \Phi_{ij,k\gamma}$ with

$$\Phi_{ij,k\gamma} := \begin{cases} (\partial_{1\gamma} + \lambda_i)^{k-1} \phi_{ij\gamma} & \text{if } k \neq d \text{ or } \gamma \neq \beta \\ (\partial_{1\alpha} + \lambda_i)^d \phi_{ij\alpha} & \text{if } k = d \text{ and } \gamma = \beta \end{cases} \tag{2.109}$$

This is also a polynomial in x; moreover we can write

$$(\partial_{1\alpha} + \lambda_i)^d \phi_{ij\alpha} = (\partial_{1\alpha} + \lambda_i)^{d-1} (\lambda_i \phi_{ij\alpha} + \partial_{1\alpha} \phi_{ij\alpha}) \tag{2.110}$$

But now $M_{\alpha\beta}$ has also a row (for $k = d - 1$, $\gamma = \alpha$) whose generic element reads $(\lambda_i + \partial_{1\alpha})^{d-1} \phi_{ij\alpha}$, and we can subtract this row multiplied by λ_1 (say) to the row (2.110) without altering the determinant, so that

$$\Phi_{ij,k\gamma} := \begin{cases} (\partial_{1\gamma} + \lambda_i)^{k-1} \phi_{ij\gamma} & \text{if } k \neq d \text{ or } \gamma \neq \beta \\ (\partial_{1\alpha} + \lambda_i)^{d-1} ((\lambda_i - \lambda_1) \phi_{ij\alpha} + \partial_{1\alpha} \phi_{ij\alpha}) & \text{if } k = d \text{ and } \gamma = \beta \end{cases}$$

This means that $\tau_{W\alpha\beta}$ is the determinant of a matrix whose generic entry is equal or of degree strictly less than the corresponding one on τ_W; it follows that the degree of $\tau_{W\alpha\beta}$ is strictly less than τ_W, and this (again by Sato's formula) implies that the off-diagonal components of $\tilde{\psi}_W$ tends to zero as $x \to \infty$. □

2.4.3 Examples

In subsection 2.3.5 we saw that the simplest (non-degenerate) solutions coming from the 1-point Grassmannian $\mathrm{Gr}_\lambda(m)$ are the ones associated to the affine cell determined by the partition (m^m), and that they are completely described by a $m \times m$ matrix α. In the present setting these solutions read simply

$$\tau(h) = \det(X_\lambda - \alpha) \qquad \tau_{\alpha\beta} = -\operatorname{cof}_{\beta,\alpha}(X_\lambda - \alpha) \tag{2.111}$$

$$\tilde{\psi}(h, z) = I_m - (X_\lambda - \alpha)^{-1} \frac{1}{z - \lambda} \tag{2.112}$$

It is then natural to consider the points of $\mathrm{Gr}^{\mathrm{Ad}}(m)$ obtained by taking a support consisting of $n > 1$ points, say $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, and choosing for each $i \in \{1, \ldots, n\}$ a point in $\mathrm{Gr}_{\lambda_i}(m)$ lying in that same affine cell; these points are the multicomponent analogues of the "simple points" of subsection 1.4.1.

Let's call $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n)$ the abstract Grassmannian coordinates of such a point and $\mathcal{W}_{\Lambda,\boldsymbol{\alpha}}$ the corresponding point in $\mathrm{Gr}^{\mathrm{Ad}}(m)$. The image of $\mathcal{W}_{\Lambda,\boldsymbol{\alpha}}$ in $\mathrm{Gr}^{\mathrm{rat}}(m)$ by the embedding (2.43) is determined as follows: first of all notice that the action of the adjoint involution on the subspace (2.89) is given by

$$W_{\lambda,\alpha}^* = W_{\lambda,-\alpha^\top} \tag{2.113}$$

(a direct generalization of formula (1.93)). Following the recipe defined in subsection 2.2.2 we see that the point $\mathcal{W}_{\Lambda,\boldsymbol{\alpha}} \in \mathrm{Gr}^{\mathrm{Ad}}(m)$ corresponds to the subspace of $\mathrm{Gr}^{\mathrm{rat}}(m)$ consisting of vectors of rational functions with at most simple poles at $\{\lambda_1, \dots, \lambda_n\}$ and satisfying the following system of mn conditions:

$$\langle f, \omega_{ij} \rangle_{\lambda_i} = 0 \qquad \text{for all } 1 \le j \le m,\ 1 \le i \le n \tag{2.114}$$

where $\omega_{ij} := (\delta_{jk} \frac{1}{z-\lambda_i} - \alpha_{kj})_{k=1\dots m}$. The resulting subspace $W_{\Lambda,\boldsymbol{\alpha}} \in \mathrm{Gr}^{\mathrm{ad}}(m)$ is again the annihilator of a space $C \subseteq \mathscr{C}^{(m)}$ generated by n systems of conditions of the form (2.90) (one for each λ_i), with coefficients a_{ijk} given by

$$a_{ijk} = -\alpha_{ikj} - \delta_{jk} \sum_{\ell \neq i} \frac{1}{\lambda_i - \lambda_\ell} \tag{2.115}$$

We conclude that tau function associated to $W_{\Lambda,\boldsymbol{\alpha}}$ is the determinant of the following block matrix:

$$\begin{pmatrix} Y_{\lambda_1} & \dots & Y_{\lambda_n} \\ \lambda_1 Y_{\lambda_1} + I & \dots & \lambda_n Y_{\lambda_n} + I \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} Y_{\lambda_1} + (n-1)\lambda_1^{n-2} I & \dots & \lambda_n^{n-1} Y_{\lambda_n} + (n-1)\lambda_n^{n-2} I \end{pmatrix}$$

where $Y_{\lambda_i} := X_{\lambda_i} + A_i^\top$ and $A_i := (a_{ijk})_{j,k=1\dots m}$, whereas the off-diagonal tau functions $\tau_{\alpha\beta}$ are obtained in the usual manner (i.e., replacing the β-th line of the bottom blocks with the α-th line of the blocks $\lambda_i^n Y_{\lambda_i} + n\lambda_i^{n-1} I$). Finally, the matrix Baker function associated to $W_{\Lambda,\boldsymbol{\alpha}}$ is easily obtained either via Sato's formula or by a direct calculation starting from the system (2.114).

We now consider in particular the case $m = n = 2$. Take a subspace $W \in \mathrm{Gr}^{\mathrm{ad}}(2)$ with support consisting of the two points $\{\lambda, \mu\}$ and, for each one of them, a point in the affine cell determined by the partition (22), described respectively by the two matrices $\alpha = (\alpha_{jk})_{j,k=1\dots 2}$ and $\beta = (\beta_{jk})_{j,k=1\dots 2}$; define finally the parameters a_{jk} and b_{jk} according to (2.115) for $i = 1, 2$. It is convenient to express the resulting tau function by grouping together the terms according to the powers of $\delta := (\mu - \lambda)^{-1}$:

$$\tau(\bar{\boldsymbol{t}}) = \tau_0(\bar{\boldsymbol{t}}) + \delta \tau_1(\bar{\boldsymbol{t}}) + \delta^2 \tau_2(\bar{\boldsymbol{t}})$$

To shorten the following expressions we introduce the notation

$$\begin{aligned} \mathcal{X}_{\gamma\lambda} := a_{\gamma\gamma} + X_{\gamma\lambda} &= a_{\gamma\gamma} + t_1^{(\gamma)} + 2t_2^{(\gamma)}\lambda + 3t_3^{(\gamma)}\lambda^2 + \dots \\ &= a_{\gamma\gamma} + \frac{1}{2}x + t_2\lambda + \frac{3}{2}t_3\lambda^2 + \dots \end{aligned}$$

We then have

$$\tau_0 = \mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda}\mathcal{X}_{1\mu}\mathcal{X}_{2\mu} - a_{12}a_{21}\mathcal{X}_{1\mu}\mathcal{X}_{2\mu} - b_{12}b_{21}\mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda} + a_{12}a_{21}b_{12}b_{21}$$

$$\tau_1 = -\mathcal{X}_{2\lambda}\mathcal{X}_{1\mu}\mathcal{X}_{2\mu} + \mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda}\mathcal{X}_{2\mu} - \mathcal{X}_{1\lambda}\mathcal{X}_{1\mu}\mathcal{X}_{2\mu} + \mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda}\mathcal{X}_{1\mu} + \\ - a_{12}a_{21}(\mathcal{X}_{1\mu} + \mathcal{X}_{2\mu}) + b_{12}b_{21}(\mathcal{X}_{1\lambda} + \mathcal{X}_{2\lambda})$$

$$\tau_3 = \mathcal{X}_{1\mu}\mathcal{X}_{2\mu} + \mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda} - \mathcal{X}_{1\lambda}\mathcal{X}_{2\mu} - \mathcal{X}_{2\lambda}\mathcal{X}_{1\mu} - (a_{12} - b_{12})(a_{21} - b_{21})$$

The reduced Baker function may be written as

$$\tilde{\psi} = I_2 - \frac{1}{\tau}W^\lambda \frac{1}{z-\lambda} - \frac{1}{\tau}W^\mu \frac{1}{z-\mu}$$

where

$$W_{11}^\lambda = -\mathcal{X}_{1\mu}\mathcal{X}_{2\lambda}\mathcal{X}_{2\mu} + b_{12}b_{21}\mathcal{X}_{2\lambda} + \delta(-2\mathcal{X}_{2\lambda}\mathcal{X}_{2\mu} + \mathcal{X}_{1\mu}\mathcal{X}_{2\mu} - \mathcal{X}_{1\mu}\mathcal{X}_{2\lambda} - (a_{12}+b_{12})b_{21}) + \\ + 2\delta^2(\mathcal{X}_{2\mu} - \mathcal{X}_{2\lambda})$$

$$W_{12}^\lambda = a_{21}\mathcal{X}_{1\mu}\mathcal{X}_{2\mu} - a_{21}b_{12}b_{21} + \delta(a_{21}(2\mathcal{X}_{2\mu} + \mathcal{X}_{1\mu}) + b_{21}\mathcal{X}_{1\lambda}) + 2\delta^2(a_{21} - b_{21})$$

$$W_{21}^\lambda = a_{12}\mathcal{X}_{1\mu}\mathcal{X}_{2\mu} - a_{12}b_{12}b_{21} + \delta(a_{12}(2\mathcal{X}_{1\mu} + \mathcal{X}_{2\mu}) + b_{12}\mathcal{X}_{2\lambda}) + 2\delta^2(a_{12} - b_{12})$$

$$W_{22}^\lambda = -\mathcal{X}_{1\lambda}\mathcal{X}_{1\mu}\mathcal{X}_{2\mu} + b_{12}b_{21}\mathcal{X}_{1\lambda} + \delta(-2\mathcal{X}_{1\lambda}\mathcal{X}_{1\mu} + \mathcal{X}_{1\mu}\mathcal{X}_{2\mu} - \mathcal{X}_{1\lambda}\mathcal{X}_{2\mu} - (a_{21}+b_{21})b_{12}) + \\ + 2\delta^2(\mathcal{X}_{1\mu} - \mathcal{X}_{1\lambda})$$

and

$$W_{11}^\mu = -\mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda}\mathcal{X}_{2\mu} + a_{12}a_{21}\mathcal{X}_{2\mu} + \delta(2\mathcal{X}_{2\lambda}\mathcal{X}_{2\mu} + \mathcal{X}_{1\lambda}\mathcal{X}_{2\mu} - \mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda} + (a_{12}+b_{12})a_{21}) + \\ + 2\delta^2(\mathcal{X}_{2\lambda} - \mathcal{X}_{2\mu})$$

$$W_{12}^\mu = b_{21}\mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda} - a_{12}a_{21}b_{21} + \delta(-b_{21}(2\mathcal{X}_{2\lambda} + \mathcal{X}_{1\lambda}) - a_{21}\mathcal{X}_{1\mu}) + 2\delta^2(b_{21} - a_{21})$$

$$W_{21}^\mu = b_{12}\mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda} - a_{12}a_{21}b_{12} + \delta(b_{12}(2\mathcal{X}_{1\lambda} + \mathcal{X}_{2\lambda}) - a_{12}\mathcal{X}_{2\mu}) + 2\delta^2(b_{12} - a_{12})$$

$$W_{22}^\mu = -\mathcal{X}_{1\lambda}\mathcal{X}_{1\mu}\mathcal{X}_{2\lambda} + a_{12}a_{21}\mathcal{X}_{1\mu} + \delta(2\mathcal{X}_{1\lambda}\mathcal{X}_{1\mu} + \mathcal{X}_{1\mu}\mathcal{X}_{2\lambda} - \mathcal{X}_{1\lambda}\mathcal{X}_{2\lambda} + (a_{21}+b_{21})a_{12}) + \\ + 2\delta^2(\mathcal{X}_{1\lambda} - \mathcal{X}_{1\mu})$$

Notice that when $\alpha = \mathrm{diag}(\alpha_{11}, \alpha_{22})$ and $\beta = \mathrm{diag}(\beta_{11}, \beta_{22})$ the tau function factorizes as $\tau = \tau_1\tau_2$ where

$$\tau_i = \mathcal{X}_{i\lambda}\mathcal{X}_{i\mu} + \delta(\mathcal{X}_{i\lambda} - \mathcal{X}_{i\mu})$$

This is exactly the tau function of a solution to the scalar (i.e. 1-component) KP equation coming from a simple point with support $\{\lambda, \mu\}$ and abstract coordinates $(\alpha_{ii}, \beta_{ii})$. If we denote by $\tilde{\psi}^{(i)}$ the corresponding reduced (scalar) Baker function, we see that the matrix Baker function may be written as

$$\psi = \begin{pmatrix} \tilde{\psi}^{(1)}\tilde{g} & 0 \\ 0 & \tilde{\psi}^{(2)}\tilde{g} \end{pmatrix}$$

i.e., the matrix Baker function associated to this particular point in $\mathrm{Gr}^{\mathrm{ad}}(2)$ is simply the direct product of two scalar Baker functions (with the same evolution parameter $\tilde{g}$).

These considerations naturally extend to more general ($m > 2$, $n > 2$) settings, so that for example we can combine 3 different solutions of the scalar KP equation with n-point support in a single solution (again with n-point support) of the 3×3 matrix KP equation, and so on.

2.4.4 The multicomponent KP/CM correspondence

Consider now a solution of the matrix KP equation of the form

$$u = -2\sum_{i=1}^{n} \frac{M_i(\boldsymbol{t}')}{(x - x_i(\boldsymbol{t}'))^2} \tag{2.116a}$$

$$w = \frac{1}{2}\sum_{i=1}^{n} \left(\frac{M_i(\boldsymbol{t}')}{(x - x_i(\boldsymbol{t}'))^3} - \frac{1}{2}\left(\frac{M_i(\boldsymbol{t}')\dot{x}_i(\boldsymbol{t}')}{(x - x_i(\boldsymbol{t}'))^2} + \frac{\dot{M}_i(\boldsymbol{t}')}{x - x_i(\boldsymbol{t}')} \right) \right) \tag{2.116b}$$

where we are using the notations $\boldsymbol{t}' = \{t_k\}_{k>1}$ and $\dot{x}_i = \partial_2 x_i$. If we substitute this ansatz in the matrix KP equation (2.99) and impose the vanishing of the residue at each (highest-order) pole we see that it must be

$$M_k(\boldsymbol{t}')^2 = M_k(\boldsymbol{t}') \qquad \text{for all } k = 1 \dots n \tag{2.117}$$

i.e., that the M_k's are projection operators on $\mathbb{C}^m$.

Suppose that these matrices have all rank 1; then for each k we can write them as

$$M_k(\boldsymbol{t}') = e_k(\boldsymbol{t}') \otimes f_k(\boldsymbol{t}') \tag{2.118}$$

This defines a set of vectors $\{e_k\}$ and a set of covectors $\{f_k\}$; notice that e_k is the unique eigenvector of M_k associated to the eigenvalue 1. Plugging this expression into (2.117) we see that $\langle f_k, e_k\rangle = 1$ for every k.

Now if we define two matrices L, P as

$$\begin{pmatrix} \frac{1}{2}\dot{x}_1 & \frac{\langle f_1,e_2\rangle}{x_1-x_2} & \dots & \frac{\langle f_1,e_n\rangle}{x_1-x_n} \\ \frac{\langle f_2,e_1\rangle}{x_2-x_1} & \frac{1}{2}\dot{x}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\langle f_n,e_1\rangle}{x_n-x_1} & \dots & \dots & \frac{1}{2}\dot{x}_n \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{\langle f_1,e_2\rangle}{(x_1-x_2)^2} & \dots & \frac{\langle f_1,e_n\rangle}{(x_1-x_n)^2} \\ \frac{\langle f_2,e_1\rangle}{(x_2-x_1)^2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\langle f_n,e_1\rangle}{(x_n-x_1)^2} & \dots & \dots & 0 \end{pmatrix} \tag{2.119}$$

then it is known (see [13, 24]) that the matrix KP equation (2.99) imply $\dot{L} = [P, L]$, i.e. the Lax equation associated to the pair (2.119). But this is exactly a Lax representation for the multicomponent Calogero-Moser system of Section 2.1, so that the evolution of the poles x_i and of the vectors e_k, f_k determined by a rank-1 solution of the form (2.116a) is given by the equations (2.3–2.5). This is the multicomponent version of the KP/CM correspondence.

We are now able to give a geometrical interpretation for this correspondence analogous to the one given by Wilson in [43] for the scalar case. Indeed let's denote by $\mathrm{Gr}^{\mathrm{ad}}(n, m)'$ the subset of the m-component adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}(m)$ made of those subspaces W whose associated operator Q_W gives a solution to the matrix KP equation of the form (2.116a); we then have maps

$$\gamma_{n,m} \colon \mathrm{Gr}^{\mathrm{ad}}(n, m)' \to \mathcal{C}'_{n,m}$$

defined by mapping W to the point $(X, Y, V, W) \in \mathcal{C}'_{n,m}$, where:

- $X = \mathrm{diag}(x_1(\mathbf{0}), \ldots, x_n(\mathbf{0}))$;
- Y is the first matrix of the Lax pair (2.119) with $\frac{1}{2}\dot{x}_i(\mathbf{0})$ on the diagonal;
- V is the $m \times n$ matrix whose rows are the $f_i(\mathbf{0})$, and
- W is the $n \times m$ matrix whose columns are the $e_i(\mathbf{0})$.

At the moment we lack an explicit formula for the map going in the opposite direction (e.g. for the tau function of the solution associated to a point of $\mathcal{C}'_{n,m}$); in the presence of such a formula, the bijectivity of the correspondence could probably be proven by methods similar to those used in subsection 1.4.2.

Appendix A

Symmetric functions

Let's denote by h_k ($k \geq 0$) the complete homogeneous symmetric functions and by p_k ($k \geq 1$) the power sum symmetric functions. Consider the associated generating functions:

$$H(z) = \sum_{k \geq 0} h_k z^k \quad \text{and} \quad P(z) = \sum_{k \geq 1} p_k z^{k-1}$$

Then we have the fundamental relation (see e.g. [26]):

$$P(z) = \frac{d}{dz} \log H(z)$$

that can be rewritten in the form

$$H(z) = \exp \int P(z)$$

By $\int P(z)$ we mean formal integration of power series. Hence we get

$$1 + h_1 z + h_2 z^2 + \cdots = \exp\left(p_1 z + \frac{1}{2} p_2 z^2 + \frac{1}{3} p_3 z^3 + \ldots\right) \tag{A.1}$$

This identity gives us a relationship between the two sets of coefficients that we used in subsection 1.2.4 to describe the elements of the groups $\Gamma_+(R)$. Remember that a function $g \in \Gamma_+(R)$ may be represented in the two different ways

$$g(z) = 1 + \sum_{k \geq 1} h_k z^k = \exp \sum_{i \geq 1} t_i z^i \tag{A.2}$$

By comparing with (A.1) we see that the relationship that holds between the coefficients $\boldsymbol{h} = \{h_k\}_{k \geq 1}$ and the coefficients $\boldsymbol{t} = \{t_k\}_{k \geq 1}$ is very similar to the one between the complete homogeneous symmetric functions and the power sum symmetric functions (it becomes exactly the same putting $it_i = p_i$). From this we deduce the relations

$$\begin{aligned} h_1 &= t_1 \\ 2! h_2 &= t_1^2 + 2t_2 \\ 3! h_3 &= t_1^3 + 6t_1 t_2 + 6t_3 \\ 4! h_4 &= t_1^4 + 12t_1^2 t_2 + 24t_1 t_3 + 12t_2^2 + 24t_4 \end{aligned} \tag{A.3}$$

with corresponding inverses

$$\begin{aligned} t_1 &= h_1 \\ 2t_2 &= 2h_2 - h_1^2 \\ 3t_3 &= 3h_3 - 3h_1h_2 + h_1^3 \\ 4t_4 &= 4h_4 - 4h_1h_3 - 2h_2^2 + 4h_1^2h_2 - h_1^4 \end{aligned} \tag{A.4}$$

and so on.

Sometimes it is useful to consider the coefficients associated to g^{-1} instead; for this reason we define the two further sequences $\boldsymbol{h}^-$ and $\boldsymbol{t}^-$ by

$$g^{-1} = 1 + \sum_{k\geq 1} h_k^- z^k = \exp \sum_{i\geq 1} t_i^- z^i \tag{A.5}$$

Then obviously $\boldsymbol{t}^- = -\boldsymbol{t}$ and the relation between $\boldsymbol{h}$ and $\boldsymbol{h}^-$ is the one that holds between mutually inverse power series:

$$h_k^- = -\sum_{j=1}^{k} h_j h_{k-j}^- \tag{A.6}$$

Explicitly

$$\begin{aligned} h_1^- &= -h_1 \\ h_2^- &= h_1^2 - h_2 \\ h_3^- &= -h_1^3 + 2h_1h_2 - h_3 \\ h_4^- &= h_1^4 - 3h_1^2h_2 + 2h_1h_3 + h_2^2 - h_4 \end{aligned} \tag{A.7}$$

These can be combined with the formulas of tables (A.3) and (A.4) to get all the possible transformations between the four families of coefficients.

For example if we take $g(z) = 1 - z$ (this function belongs to $\Gamma_+(R)$ for every $R < 1$) then we get $\boldsymbol{h} = \{-1, 0, 0, \dots\}$, $\boldsymbol{h}^- = \{1, 1, 1, \dots\}$, $\boldsymbol{t} = \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$ and $\boldsymbol{t}^- = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.

Appendix B

Partial fraction decomposition

Let f be a rational function on $\mathbb{C}P^1$. This means that there exist polynomials p and q with $q \neq 0$ such that

$$f(z) = \frac{p(z)}{q(z)} \quad \text{for all } z \in \operatorname{dom} f \tag{B.1}$$

where $\operatorname{dom} f := \{\, z \in \mathbb{C}P^1 \mid q(z) \neq 0 \,\}$ is the (maximal) *domain of definition* of f. By the usual division algorithm we can determine a polynomial p_∞ (possibly zero) and a polynomial r with $\deg r < \deg q$ such that

$$f(z) = p_\infty(z) + \frac{r(z)}{q(z)} \tag{B.2}$$

(This is the direct sum decomposition $\mathcal{R} = \mathcal{P} \oplus \mathcal{R}_-$ used from Section 1.2 onward.) The rational function r/q being analytic at $\lambda_0 := \infty$, it follows that the principal part of f at that point is $s_0(z) := p_\infty(z) - p_\infty(0)$; notice that $p_\infty(0) = f(\infty)$.

Now denote by $\lambda_1, \ldots, \lambda_n$ the (distinct) zeros of q in $\mathbb{C}$ and by $m_1, \ldots, m_n$ the corresponding multiplicities. Let s_i be the principal part of f at λ_i:

$$s_i(z) = \sum_{j_i=1}^{m_i} \frac{a_{j_i}}{(z - \lambda_i)^{j_i}} \tag{B.3}$$

Then we have the *partial fraction decomposition* (see e.g. [17, §4.4]):

$$\frac{r(z)}{q(z)} = \sum_{i=1}^{n} s_i(z) \tag{B.4}$$

Putting all together we have the following:

Theorem B.5. *A rational function f is equal to the sum of its principal parts at the poles (possibly including ∞) and its value at infinity:*

$$f(z) = f(\infty) + \sum_{i=0}^{n} s_i(z) \tag{B.6}$$

It is now very easy to describe the embedding $\mathcal{R} \to \mathbb{C}((z^{-1}))$ given by Laurent expansion at infinity. We first recall that in any ring of formal power series we have

$$\frac{1}{1-X} = \sum_{k\geq 0} X^k$$

and consequently in $\mathbb{C}((z^{-1}))$

$$\frac{1}{z-\lambda} = z^{-1}\frac{1}{1-\frac{\lambda}{z}} = \sum_{k\geq 0} \lambda^k z^{-1-k} = \frac{1}{z} + \frac{\lambda}{z^2} + \frac{\lambda^2}{z^3} + \dots$$

for any $\lambda \in \mathbb{C}$. So if f is a rational function with n simple poles at $\lambda_1, \dots, \lambda_n$ we have

$$f = f(\infty) + \sum_{i=1}^{n} \frac{a_i}{z-\lambda_i} = f(\infty) + \sum_{k\geq 0} \left(\sum_{i=1}^{n} a_i \lambda_i^k \right) z^{-1-k}$$

Consider now a generic pole of order m and the corresponding principal part (B.3). Using the well-known hypergeometric series expansion (see e.g. [14])

$$F(a;z) = \sum_{k\geq 0} \frac{a^{\overline{k}}}{k!} z^k = \frac{1}{(1-z)^a}$$

(where $a^{\overline{k}} := \frac{(a+k+1)!}{(a-1)!}$ is the *rising factorial power*) we can write

$$\frac{1}{(z-\lambda)^a} = \sum_{k\geq 0} \frac{a^{\overline{k}}}{k!} \lambda^k z^{-a-k}$$

so that equation (B.3) becomes

$$s_i(z) = \sum_{j_i=1}^{m_i} a_{j_i} \sum_{k\geq 0} \frac{j_i^{\overline{k}}}{k!} \lambda_i^k z^{-j_i-k} \tag{B.7}$$

Then equation (B.6) reads

$$f(z) = f(\infty) + \sum_{k\geq 0} \sum_{i=0}^{n} \sum_{j_i=1}^{m_i} a_{j_i} \frac{j_i^{\overline{k}}}{k!} \lambda_i^k z^{-j_i-k} \tag{B.8}$$

and this gives the afore-mentioned embedding in an explicit form.

Appendix C

Schubert cells

In this Appendix we recall the classical decomposition of finite-dimensional Grassmannians in Schubert cells. Our approach follows [15, 28].

Let $\mathrm{Gr}(k, V)$ denote the Grassmannian of k-dimensional subspaces in a n-dimensional complex vector space V; it is a (complex) smooth manifold of dimension $k(n-k)$, connected and compact. If we fix once and for all an ordered basis $E = (e_1, \dots, e_n)$ for V then every $W \in \mathrm{Gr}(k, V)$ may be represented by a $k \times n$ matrix of maximum rank (i.e., rank k) such that its rows are a basis for W; such a matrix is determined up to multiplication from the left by a $k \times k$ invertible matrix.

Now consider the complete flag determined by E:

$$V_i := \mathrm{span}\{e_1, \dots, e_i\}$$

For every $W \in \mathrm{Gr}(k, n)$ we have the non-decreasing sequence of subspaces

$$0 = W \cap V_0 \subseteq W \cap V_1 \subseteq \cdots \subseteq W \cap V_{n-1} \subseteq W \cap V_n = W$$

If W is in generic position with respect to E then the intersections $W \cap V_i$ will be zero for $i \leq n-k$ and they will have dimension $i - (n-k)$ henceforth.

Generally, for every $W \in \mathrm{Gr}(k, n)$ we put $d_i := \dim(W \cap V_i)$; then we have the non-decreasing sequence of natural numbers

$$0 = d_0 \leq d_1 \leq \cdots \leq d_{n-1} \leq d_n = k$$

Two consecutive integers in this sequence differ by at most 1, so it contains exactly k "jumps". The **Schubert symbol** of W is the sequence of k natural numbers for which there is a jump, i.e. the indices i such that $d_i = d_{i-1} + 1$. Clearly Schubert symbols are increasing sequences of numbers between 1 and n; given any sequence $\sigma = (\sigma_1, \dots, \sigma_k)$ of this type, the associated **Schubert cell** C_σ is the set of subspaces $W \in \mathrm{Gr}(k, n)$ which have σ as Schubert symbol.

Now take a subspace $W \in C_\sigma$; we can build a canonical basis for it as follows. By definition of Schubert symbol, $W \cap V_{\sigma_1}$ has dimension 1; let v_1 be a generator of this space normalized such that $\langle v_1, e_{\sigma_1}\rangle = 1$, or in other words

$$v_1 = (\underbrace{*,\dots,*}_{\sigma_1-1},1,0,\dots,0)$$

Then take v_2 such that $\{v_1, v_2\}$ generates $W \cap V_{\sigma_2}$, normalized in such a manner that $\langle v_2, e_{\sigma_1}\rangle = 0$ and $\langle v_2, e_{\sigma_2}\rangle = 1$, i.e.

$$v_2 = (\underbrace{*,\dots,*}_{\sigma_1-1},0,\underbrace{*,\dots,*}_{\sigma_2-\sigma_1-1},1,0,\dots,0)$$

Continuing the process all the way to v_k we obtain a basis for W such that the corresponding representative matrix is

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = \begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots \end{pmatrix}$$

Vice versa, every matrix of this form describes a k-dimensional space that belongs to C_σ. Now, such a matrix has exactly[1]

$$d_\sigma := \sum_{i=0}^{k-1}(k-i)(\sigma_{i+1}-\sigma_i-1) \tag{C.1}$$

free elements; we thus have homeomorphisms

$$C_\sigma \cong \mathbb{C}^{d_\sigma}$$

and it can be shown that the sets C_σ give a cell decomposition for $\mathrm{Gr}(k,n)$; it generalizes the well-known cell structure of projective spaces,

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^1 \cup \mathbb{C}^0$$

to which it reduces for $k = 1$.

Given a Schubert symbol σ the **partition associated to** σ is defined by

$$p(\sigma) := (\sigma_k - k, \dots, \sigma_1 - 1) \tag{C.2}$$

Clearly it is a partition of length not greater than k and whose parts are not greater than $n-k$, and the correspondence (C.2) between Schubert symbols and partitions of this form is bijective. It is not difficult to show that the number d_σ defined by (C.1) is precisely the weight of $p(\sigma)$, so we conclude that the number of d-dimensional cells in $\mathrm{Gr}(k,V)$ is exactly equal to the number of partitions of n of length not greater than k with parts not greater than $n-k$.

For example let's consider the cell structure of the first nontrivial (i.e. not isomorphic to a projective space) Grassmannian, $\mathrm{Gr}(2,4)$. The possible Schubert symbols, with corresponding partitions, are

[1] Here we assume $\sigma_0 = 0$.

d_0	d_1	d_2	d_3	d_4	σ	$p(\sigma)$	$\dim C_\sigma$
0	0	0	1	2	(34)	(22)	4
0	0	1	1	2	(24)	(21)	3
0	0	1	2	2	(23)	(11)	2
0	1	1	1	2	(14)	(2)	2
0	1	1	2	2	(13)	(1)	1
0	1	2	2	2	(12)	()	0

The cell of maximal dimension $C_{(34)}$ contains exactly the subspaces in generic position with respect to the chosen basis.

Bibliography

[1] H. Airault, H. P. McKean, and J. Moser. Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem. *Commun. Pure Appl. Math.*, 30:95–148, 1977.

[2] O. Babelon, D. Bernard, and M. Talon. *Introduction to Classical Integrable Systems.* Cambridge University Press, 2003.

[3] V. Baranovsky, V. Ginzburg, and A. Kuznetsov. Quiver varieties and a noncommutative $\mathbb{P}^2$. *Compositio Math.*, 134(3):283–318, 2002. `arXiv:math/0103068`.

[4] V. Baranovsky, V. Ginzburg, and A. Kuznetsov. Wilson's Grassmannian and a noncommutative quadric. *Int. Math. Res. Not.*, (21):1155–1197, 2003. `arXiv:math/0203116`.

[5] D. Ben-Zvi and T. Nevins. $\mathcal{D}$-bundles and integrable hierarchies. Preprint `arXiv:math/0603720`, 2006.

[6] D. Ben-Zvi and T. Nevins. From solitons to many-body systems. *Pure Appl. Math. Q.*, 4(2, part 1):319–361, 2008. `arXiv:math/0310490`.

[7] N. Bourbaki. *Theory of Sets.* Hermann, 1968.

[8] F. Calogero. Ground state of a one-dimensional n-body system. *J. Math. Phys.*, 10(12):2197–2200, 1969.

[9] F. Calogero. Solution of a three-body problem in one dimension. *J. Math. Phys.*, 10(12):2191–2196, 1969.

[10] F. Calogero. Solution of the one-dimensional n-body problems with quadratic and/or inversely quadratic pair potentials. *J. Math. Phys.*, 12:419–436, 1971.

[11] L. A. Dickey. On Segal-Wilson's definition of the τ-function and hierarchies AKNS-D and mcKP. In Olivier Babelon, Pierre Cartier, and Yvette Kosmann-Schwarzbach, editors, *Integrable Systems. The Verdier Memorial Conference*, pages 147–162. Birkhäuser, 1993.

[12] L. A. Dickey. *Soliton Equations and Hamiltonian Systems.* World Scientific, 2nd edition, 2003.

[13] J. Gibbons and T. Hermsen. A generalization of the Calogero-Moser system. *Physica*, 11D:337–348, 1984.

[14] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, 2nd edition, 1994.

[15] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley-Interscience, 1978.

[16] M. Hamanaka. Commuting flows and conservation laws for noncommutative Lax hierarchies. *J. Math. Phys.*, 46(5):052701, 13, 2005. `arXiv:hep-th/0311206v2`.

[17] P. Henrici. *Applied and Computational Complex Analysis*, volume 1. Wiley-Interscience, 1974.

[18] V. G. Kac and J. W. van de Leur. The n-component KP hierarchy and representation theory. *J. Math. Phys.*, 44:3245–3293, 2003.

[19] B. B. Kadomtsev and V. J. Petviashvili. On the stability of solitary waves in weakly dispersive media. *Soviet Phys. Dokl.*, 15:539–541, 1970.

[20] A. Kasman. Bispectral KP solutions and linearization of Calogero-Moser particle system. *Commun. Math. Phys.*, 172:427–448, 1995.

[21] D. Kazhdan, B. Kostant, and S. Sternberg. Hamiltonian group actions and dynamical systems of Calogero type. *Comm. Pure Appl. Math.*, 31:481–507, 1978.

[22] I. M. Krichever. Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of n particles on a line. *Funct. Anal. Appl.*, 12(1):59–61, 1978.

[23] I. M. Krichever. Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles. *Funct. Anal. Appl.*, 14(4):282–290, 1980.

[24] I. M. Krichever, O. Babelon, E. Billey, and M. Talon. Spin generalization of the Calogero-Moser system and the matrix KP equation. In *Topics in topology and mathematical physics*, volume 170 of *Amer. Math. Soc. Transl. Ser. 2*, pages 83–119. Amer. Math. Soc., 1995. `arXiv:hep-th/9411160`.

[25] M. D. Kruskal. The Korteweg-de Vries equation and related evolution equations. In A. L. Snyder Jr., editor, *Nonlinear wave motion*, pages 61–83. Amer. Math. Soc., 1974.

[26] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, 2nd edition, 1995.

[27] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry*. Springer, 2nd edition, 1999.

[28] J. W. Milnor and J. D. Stasheff. *Characteristic Classes.* Princeton University Press, 1974.

[29] J. Moser. Three integrable Hamiltonian systems connected with isospectral deformations. *Adv. Math.*, 16:1–23, 1975.

[30] M. A. Olshanetsky and A. M. Perelomov. Explicit solution of the Calogero model in the classical case and geodesic flows of zero curvature. *Lett. Nuovo Cim.*, 16:333–339, 1976.

[31] M. A. Olshanetsky and A. M. Perelomov. Classical integrable finite-dimensional systems related to Lie algebras. *Phys. Rep.*, 71(5):313–400, 1981.

[32] M. A. Olshanetsky and A. M. Perelomov. Quantum integrable systems related to Lie algebras. *Phys. Rep.*, 94(6):313–404, 1983.

[33] D. V. Osipov. Krichever correspondence for algebraic varieties. *Izv. Math.*, 65(5):941–975, 2001. `arXiv:math/0003188`.

[34] A. N. Parshin. Krichever correspondence for algebraic surfaces. Preprint `arXiv:math/9911097`, 1999.

[35] A. N. Parshin. On a ring of formal pseudo-differential operators. *Proc. Steklov Math. Institute*, 224:266–280, 1999. `arXiv:math/9911098`.

[36] A. M. Perelomov. *Integrable systems of classical mechanics and Lie algebras.* Birkhäuser, 1990.

[37] A. Pressley and G. Segal. *Loop Groups.* Clarendon Press, 1986.

[38] M. Sato and Y. Sato. Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold. *RIMS Kokyuroku*, 439:30–46, 1981.

[39] G. Segal and G. Wilson. Loop groups and equations of KdV type. *Publ. Math. I.H.E.S.*, 61:5–65, 1985.

[40] T. Shiota. Calogero-Moser hierarchy and KP hierarchy. *J. of Math. Phys.*, 35:5844–5849, 1994. `arXiv:hep-th/9402021`.

[41] W. R. Thickstun. A system of particles equivalent to solitons. *J. Math. Anal. Appl.*, 55:335, 1976.

[42] G. Wilson. Bispectral commutative ordinary differential operators. *J. reine angew. Math.*, 442:177–204, 1993.

[43] G. Wilson. Collisions of Calogero-Moser particles and an adelic Grassmannian. *Invent. Math.*, 133:1–41, 1998.

Acknowledgements

I am very grateful to all the people I interacted with during the making of this work; however some of them clearly deserve a special mention.

First of all I wish to thank my advisor Claudio Bartocci for his patience, his constant encouragement and a host of useful conversations. For much the same reasons I also owe a lot to Igor Mencattini, who suggested me the beautiful subject of this Thesis and with whom I was very happy to work (regrettably, for a too short amount of time).

I would like to thank the referee, Prof. Volodya Roubtsov, for his useful comments, as well as Marco Pedroni and Giovanni Ortenzi for kindly taking a sincere interest in my work and providing new ideas for its development.

At an early stage in the thinking of this Thesis I have greatly benefited from a four months stay at the Mathematics Department in Oxford thanks to a grant provided by the ENIGMA European Research Training Network, and I am happy to take this opportunity to thank Lionel Mason, who was my advisor there, and George Wilson for many helpful discussions.

Finally I wish to thank all of my friends at the Physics and Mathematics Departments of the University of Genoa for the many happy moments that I had with them[2]. Lately I have come to see this work as the final achievement of a very significant portion of my life, and I find I have been rather lucky to spend it with them.

Of course this Thesis could not have been completed without the love and support of my family and especially my parents Giovanna and Francesco; the only reason I do not dedicate this work to them is that I feel it doesn't quite live up to.

[2]But this time I will avoid jokes about their beauty and/or niceness.